EQUALITY OF SEVEN FUNDAMENTAL SETS CONNECTED WITH $A(K)$: ANALYTIC CAPACITY-FREE PROOFS

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ABSTRACT. Let K be any compact set in $\mathbb{C}$ with connected complement, let $A(K)$ be the uniform algebra of all complex functions continuous on K and holomorphic on $\operatorname{int}(K)$, let ∂K be the topological boundary of K, let $z \in K$, and let M_z be the maximal ideal of functions in $A(K)$ vanishing at z. Using only facts from classical complex analytic function theory, and without using any results from the theory of analytic capacity, we prove the following: $z \in \partial K$ iff z is a peak point for $A(K)$ iff z belongs to the Shilov boundary of $A(K)$ iff z belongs to the Bishop minimal boundary of $A(K)$ iff M_z has a bounded approximate identity iff z satisfies the Bishop $\frac{1}{4} - \frac{3}{4}$ property iff z is a strong boundary point for $A(K)$. More specifically, the only results used in all proofs come from classical analytic function theory, properties of open connected sets in the complex plane, the Caratherodory extension theorem, the Riemann mapping theorem, Euler's formula, Rudin's estimates on finite complex products, properties of linear fractional transformations, the α-th root function in the complex plane, some new and striking Jordan curve constructions (Jordan kissing paths), and the Cohen Factorization theorem. No results are used from the theory of analytic capacity, the theory of representing or annihilating measures, Dirichlet algebra theory, Choquet boundary theory, or the Walsh-Lebesgue theorem.

1. Introduction

In [1], the following was established. Let $K \subset \mathbb{C}$ be compact, let $P(K)$ be the uniform closure in $C(K)$ of all polynomial functions restricted to K, and let $A(K)$ be the uniform algebra of all functions continuous on K and holomorphic on $\operatorname{int}(K)$. It is known (Mergelyan) that $P(K) = A(K)$ if and only if $\mathbb{C}\backslash K$ is connected. (The implication $P(K) = A(K)$ implies $\mathbb{C}\backslash K$ is connected is proved in [6] without using results from the

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theory of analytic capacity; for the converse implication, Mergelyan's original proof depended on results from the theory of analytic capacity, but the proofs given in[6] (Stout), [7] (F. Browder), [4] (Rudin) and [8] (Carleson) do not use analytic capacity theory). Let $\mathcal{K} = \{K \subset \mathbb{C}$: K is compact and $\mathbb{C} \setminus K$ is connected$\}$, let ∂K be the topological boundary of K and let $\mathcal{P}(P(K)) = \{z \in K$: there exists $f \in P(K)$ such that $f(z) = 1$, $|f(w)| < 1$ for $w \in K \setminus \{z\}\}$, i.e., the set of peak points relative to $P(K)$. By the maximum modulus principle, $\mathcal{P}(P(K)) \subseteq \partial K$. A long standing problem is to determine whether $\mathcal{P}(P(K)) = \partial K$ for all $K \in \mathcal{K}$. The rest of this paper contains the details of such a proof.

Definition. In what follows, various types of topological boundary points are used:

(1) $(\partial K)_{\mathrm{I}} = \{z_0 \in \partial K$: every open ngbd of z_0 intersects $\mathrm{int}(K)\}$ (denoted type I);

(2) $((\partial K)_{\mathrm{II}} = \{z_0 \in \partial K$: some open ngbd of z_0 does not intersect $\mathrm{int}(K)\}$ (denoted type II);

(3) $z \in \partial K$ is *circularly accessible (ca)* iff there is a $w \in \mathbb{C} \setminus K$, an $r > 0$, and a closed disk $\overline{D}(w; r) = \{u \in \mathbb{C} : |u - w| \leq r\}$ such that $\overline{D}(w; r) \cap K = \{z\}$ and $\overline{D}(w; r) \setminus \{z\} \subset \mathbb{C} \setminus K$;

(4) $z \in \partial K$ is *segmentally accessible (sa)* iff there is a segment $[z, w] = \{u \in \mathbb{C} : u = z + t(w - z), 0 \leq t \leq 1\}$ such that $[z, w] \setminus \{z\} \subset \mathbb{C} \setminus K$;

(5) $z \in \partial K$ is a *Jordan escape point (Je)* iff there exists a Jordan curve, Γ_z, initiating and terminating at z such that $K \setminus \{z\}$ is contained in the bounded component of Γ_z (which is simply-connected and connected) and such that the winding number of Γ_z w.r.t. $K \setminus \{z\}$ is $+1$;

(6) $z \in \partial K$ is an *escape point (ep)* iff there is a continuous one-to-one function $f : [0, 1] \mapsto \mathbb{C}$ such that $f(0) = z$ and $f((0, 1]) \subset \mathbb{C} \setminus K$;

(7) $\mathcal{P}_{\mathrm{ca}}(K)$, $\mathcal{P}_{\mathrm{sa}}(K)$, $\mathcal{P}_{\mathrm{Je}}(K)$, $\mathcal{P}_{\mathrm{ep}}(K)$ denote the sets of ca, sa, Je and ep points of ∂K, respectively.

Fact. *In* [1]*, the following facts are proved;*

(i) $(\partial K)_{II} \subseteq \mathcal{P}(P(K))$ *(*[1, THEOREM 5.1.ii]*).*

(ii) *If* $\mathrm{int}(K) = \emptyset$*, then* $K = (\partial K)_{II} = \mathcal{P}(P(K))$*;*

(iii) $\mathcal{P}_{ca}(K) \subseteq \mathcal{P}_{sa}(K) \subseteq \mathcal{P}_{Je}(K) = \mathcal{P}_{ep}(K) \subseteq \mathcal{P}(P(K)) \subseteq \partial K$ *and* $\mathcal{P}_{ca}(K)$ *is dense in* ∂K*.*

(iv) *If* $\operatorname{int}(K) \neq \emptyset$ *and if* $\mathcal{P}_{ca}(K) = (\partial K)_I$, *or* $\mathcal{P}_{sa}(K) = (\partial K)_I$, *or* $\mathcal{P}_{Je}(K) = (\partial K)_I$, *or* $\mathcal{P}_{ep}(K) = (\partial K)_I$, *then* $\mathcal{P}(P(K)) = \partial K$, *i.e., every boundary point is a peak point.*

(v) *If* $\operatorname{int}(K) \neq \emptyset$ *and if* $\partial(\operatorname{int}(K))$ *is Jordan curve, then every point of* ∂K *is a peak point for* $P(K)$ *(*[1, THEOREM 5.5]*).*

Facts (i) through (v) were proved without using any results from the theory of analytic capacity.

Let $\mathcal{K}^*$ be the set of $K \in \mathcal{K}$ such that: $\operatorname{int}(K) \neq \emptyset$ and there is a $z \in (\partial K)_{\mathrm{I}}$ that is not an ep (equivalently, Je) point. For $K \in \mathcal{K}^*$, the proof that each non-ep (or non-Je) point of $(\partial K)_{\mathrm{I}}$ is a peak point uses the Curtis Peak Point Criterion [1, 5] whose proof requires results from the theory of analytic capacity.

Thus, [1] contains the complete proof that: $\mathcal{P}(P(K)) = \partial K$ for all $K \in \mathcal{K}$.

We now proceed with the new capacity-free proofs of the results stated in the Abstract.

2. Notations and properties of the α-th root Function

The α-th root function is an essential tool for the results proved in what follows. The properties listed below are standard results from complex variable theory.

For $r > 0$ and $z_0 \in \mathbb{C}$, the closed disk, $\{z \in \mathbb{C} : |z - z_0| \leq r\}$, is denoted by $\overline{D}(z_0; r)$; the open disk, $\{z \in \mathbb{C} : |z - z_0| < r\}$, is denoted by $D(z_0; r)$; the circle, $\{z \in \mathbb{C} : |z - z_0| = r\}$, is denoted $C(z_0; r)$; the topological boundary of a set K is denoted by ∂K; clearly, $\partial \overline{D}(z_0; r) = \partial D(z_0; r) = C(z_0; r)$.

Definition 2.1. We define several basic concepts and their properties in the following list:

- The *exponential function* on $\mathbb{C}$, $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, is an entire function; its restriction to the infinite open strip,

$$S_{(-\pi,\pi)} = \{z = x + iy \in \mathbb{C} : (-\infty < x < \infty), (-\pi < y < \pi)\}$$

  is a conformal map from $S_{(-\pi,\pi)}$ onto $\mathbb{C} \setminus (-\infty, 0]$ with conformal inverse, $\log := \exp^{-1}$, that maps $\mathbb{C}(-\infty, 0]$ onto $S_{(-\pi,\pi)}$. For any z_1 and z_2 in $S_{(-\pi,\pi)}$, $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$, and for any $z_1$and z_2 in $\mathbb{C} \setminus (-\infty, 0]$, $\log(z_1 z_2) = \log(z_1) + \log(z_2)$.
- The *argument function*, arg, is defined for $z \in \mathbb{C} \setminus (-\infty, 0]$ by $\arg(z) = \operatorname{Im}(\log(z))$, and is continuous.

- The *α-th root function*, $\mathbb{Z}^\alpha$, for $\alpha \in (0,1]$, is defined by $\mathbb{Z}^\alpha(z) \equiv (\exp \circ m_\alpha \circ \log)(z)$, for $z \in \mathbb{C} \setminus (-\infty, 0]$, and $\mathbb{Z}^\alpha(0) = 0$, where $m_\alpha(z) = \alpha z$, for $z \in \mathbb{C}$. In other words, $\mathbb{Z}^\alpha = \exp \circ m_\alpha \circ \log$. Now $\mathbb{Z}^\alpha$ is conformal on $\mathbb{C} \setminus (-\infty, 0]$, and is a homeomorphism on $\mathbb{C}(-\infty, 0)$.
- If we define the *open cone*,
$$C_\alpha \equiv \{w \in \mathbb{C} \setminus (-\infty, 0] : |\arg(w)| < \alpha\pi\},$$
then $\mathbb{Z}^\alpha$ maps $\mathbb{C}\setminus(-\infty,0]$ conformally onto C_α, $(\mathbb{Z}^\alpha)^{-1}$ maps C_α conformally onto $\mathbb{C}\setminus(-\infty,0]$, $\mathbb{Z}^\alpha$maps $\mathbb{C}\setminus(-\infty,0)$ homeomorhically onto $\{0\}\cup C_\alpha$, and $(\mathbb{Z}^\alpha)^{-1}$ maps $\{0\}\cup C_\alpha$ homeomorphically onto $\mathbb{C} \setminus (-\infty, 0)$.
- Note that
$$\begin{aligned}(\mathbb{Z}^\alpha)^{-1} &= (\exp \circ m_\alpha \circ \log)^{-1}\\ &= (\log)^{-1} \circ (m_\alpha)^{-1} \circ (\exp)^{-1}\\ &= \exp \circ m_{\frac{1}{\alpha}} \circ \log\end{aligned}$$
is defined on C_α and $(\mathbb{Z}^\alpha)^{-1}(0) = 0$.
- Note that $C_1 = \mathbb{C} \setminus (-\infty, 0]$.
- For each $z \in \mathbb{C} \setminus (-\infty, 0]$, we have the unique representation, $z = |z| \exp(i(\arg(z)))$, where $-\pi < \arg(z) < \pi$. Furthermore, $\log(z) = \log(|z|) + i(\arg(z))$.
- For $\alpha \in (0,1]$ and $z \in \mathbb{C} \setminus (-\infty, 0)$, $\mathbb{Z}^\alpha(z) = 0$ iff $z = 0$, and for $z \neq 0$, $\mathbb{Z}^\alpha(z) = |z|^\alpha \exp(i\alpha(\arg(z)))$.
- For $\alpha \in (0,1]$ and $z \in C_\alpha$, $(\mathbb{Z}^\alpha)^{-1}(z) = 0$ iff $z = 0$, and for $z \neq 0$, $(\mathbb{Z}^\alpha)^{-1}(z) = |z|^{\frac{1}{\alpha}} \exp(i\frac{1}{\alpha}\arg(z))$.
- For $0 \leq \rho$ and $\rho < r$, define the *open deleted annulus*,
$$\begin{aligned}&\mathrm{And}(0;\rho,r)\\ &\equiv \{z \in \mathbb{C} \setminus \{0\} : \rho < |z| < r, -\pi < \arg(z) < \pi\}\\ &\subset \mathbb{C} \setminus (-\infty, 0]\end{aligned}$$
- For each $0 < \delta < 1$ and $0 < \theta_0 < \pi$, define *the arc along the circle*, $C(0;\delta)$, from $\delta\exp(-i\theta_0)$ to $\delta\exp(i\theta_0)$ by
$$\begin{aligned}&\mathrm{arc}[\delta\exp(-i\theta_0), \delta\exp(i\theta_0)]\\ &= \{\delta\exp(i\theta) : -\theta_0 \leq \theta \leq \theta_0\}.\end{aligned}$$
For each $\alpha \in (0,1]$,
$$\begin{aligned}&\mathbb{Z}^\alpha(\mathrm{arc}[\delta\exp(-i\theta_0), \delta\exp(i\theta_0)])\\ &= \mathrm{arc}[\delta^\alpha\exp(-i\alpha\theta_0), \delta\exp(i\alpha\theta_0)]\end{aligned}$$
Furthermore, as $\alpha \searrow 0$, $\delta^\alpha \nearrow 1$.

Remark. For later use in the construction of infinite products of functions in $P(K)$, we need to examine the image of $(\overline{D}(\frac{1}{2};\ \frac{1}{2})$ under the map $\mathbb{Z}^\alpha$.

Lemma 2.2. ("Teardrop" Lemma)
For $\alpha \in (0,1)$:

(2.2.1) $\mathbb{Z}^\alpha((\overline{D}(\frac{1}{2};\frac{1}{2}))$ *is compact.*

(2.2.2) $\mathbb{Z}^\alpha(\partial(\overline{D}(\frac{1}{2};\frac{1}{2})))$ *a Jordan curve.*

(2.2.3) $\mathbb{Z}^\alpha(\partial(\overline{D}(\frac{1}{2};\frac{1}{2}))) = \{0\} \cup \left\{(\cos\theta)^\alpha \exp(i\alpha\theta) : -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$. *We put $t_\alpha \equiv \mathbb{Z}^\alpha(\partial(\overline{D}(\frac{1}{2};\frac{1}{2})))$. The curve t_α, looks like the boundary of a teardrop.*

(2.2.4) $t_\alpha = \mathbb{Z}^\alpha(\partial(\overline{D}(\frac{1}{2};\frac{1}{2})) \subset \{0,1\} \cup (C_{\alpha/2} \cap D(\frac{1}{2};\frac{1}{2}))$.

(2.2.5) $\mathrm{int} t_\alpha = \mathrm{int}(\mathbb{Z}^\alpha((\overline{D}(\frac{1}{2};\frac{1}{2})))$ *is open and simply-connected and* $t_\alpha \cup \mathrm{int} t_\alpha \subset \{0,1\} \cup (C_{\alpha/2} \cap D(\frac{1}{2};\frac{1}{2}))$. *We put $T_\alpha \equiv t_\alpha \cup \mathrm{int} t_\alpha$ and call it an α-*teardrop.

(2.2.6) *For all $0 < \rho < \frac{1}{2}$ and all $0 < \delta < \frac{1}{2}$, there exists $0 < \alpha_{\rho,\delta} < 1$ such that for all $0 < \alpha \leq \alpha_{\rho,\delta}$, $\mathbb{Z}^\alpha$ maps $\overline{D}(\frac{1}{2};\frac{1}{2}) \setminus \overline{D}(0;\rho)$ into $D(1;\delta)$.*

(2.2.7) *For all $0 < \theta < \frac{\pi}{2}$, there exists $0 < \alpha_{\rho,\delta,\theta} \leq \alpha_{\rho,\delta}$ such that for all $0 < \alpha \leq \alpha_{\rho,\delta,\theta}$, (2.2.6) holds and $|\arg(z)| < \theta$ for every $z \in T_\alpha$.*

Proof. (2.2.1) $\mathbb{Z}^\alpha$ is a homeomorphism on $\mathbb{C} \setminus (-\infty, 0)$, so $\mathbb{Z}^\alpha((\overline{D}(\frac{1}{2};\frac{1}{2}))$ is compact.

(2.2.2) The function $h : \mathbb{T} \to \partial(\overline{D}(\frac{1}{2};\frac{1}{2})))$, defined by $h(z) = \frac{1}{2}(1+z)$, $z \in \mathbb{T}$, is a homeomorphism, so $\partial(\overline{D}(\frac{1}{2};\frac{1}{2}))$ is a Jordan curve. Since $\mathbb{Z}^\alpha$ is a homeomorphism, and since a homeomorphic images of a Jordan curve is also a Jordan curve, the result follows.

(2.2.3) For $0 \in \partial(\overline{D}(\frac{1}{2};\frac{1}{2}))$, $\mathbb{Z}^\alpha(0) = 0$. For $z \in \partial(\overline{D}(\frac{1}{2};\frac{1}{2})) \setminus \{0\}$, $z = (\cos\theta)\exp(i\theta)$, where $\theta = \arg(z) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This follows from the well-known fact that for any three distinct points on a circle, two of which are at the opposite ends of a diameter line, the triangle formed by using these points as vertices is a right triangle. Thus,

$$\begin{aligned} \mathbb{Z}^\alpha(z) &= \mathbb{Z}^\alpha((\cos\theta)\exp(i\theta)) \\ &= |\cos\theta|^\alpha \exp(i\alpha\theta) \\ &= (\cos\theta)^\alpha \exp(i\alpha\theta) \end{aligned}$$

Thus, (2.2.3) is proved.

(2.2.4) We know $\mathbb{Z}^\alpha(0) = 0$. For $z \in \partial(\overline{D}(\frac{1}{2}; \frac{1}{2})) \setminus \{0\}$, we proceed as follows. Define f_α: $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \partial(\overline{D}(\frac{1}{2}; \frac{1}{2})) \setminus \{0\}$ by $f_\alpha(\theta) = (\cos \alpha\theta) \exp(i\alpha\theta)$. This map is one-to-one, continuous and onto. To show that $\mathbb{Z}^\alpha((\partial(\overline{D}(\frac{1}{2}; \frac{1}{2})) \setminus \{0\}) \subset D(\frac{1}{2}; \frac{1}{2})$, it suffices to show that $(\cos \theta)^\alpha < \cos \alpha\theta$, for $0 \leq \theta < \frac{\pi}{2}$, or, equivalently, that $g(\theta) := \frac{(\cos\ \theta)^\alpha}{\cos\ \alpha\theta} < 1$ for $0< \theta < \frac{\pi}{2}$. The latter is true if $g'(\theta) < 0$ for $0< \theta < \frac{\pi}{2}$. But a straightforward calculation shows that $g'(\theta) = \frac{-\ \alpha(\cos\ \theta)^\alpha \sin((1\ -\ \alpha)\theta)}{(\cos\ \theta)(\cos\ \alpha\theta)^2} < 0$ since every term (except $-\alpha$) in the expression is positive for $0< \theta < \frac{\pi}{2}$, and $-\alpha < 0$. Since $g(\theta) = g(-\theta)$, the result also holds for $-\frac{\pi}{2} < \theta < 0$. Finally, we show that $\mathbb{Z}^\alpha((\partial(\overline{D}(\frac{1}{2}; \frac{1}{2})) \setminus \{0\}) \subset C_{\alpha/2}$. Since $\mathbb{Z}^\alpha(\partial(\overline{D}(\frac{1}{2}; \frac{1}{2}))) = \{0\} \cup f_\alpha((-\frac{\pi}{2}, \frac{\pi}{2}))$, we immediately see that $\arg(f_\alpha(\theta)) = \arg((\cos \alpha\theta) \exp(i\alpha\theta)) = \alpha\theta$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, hence $\alpha\theta \in (-\frac{\alpha}{2}\pi, \frac{\alpha}{2}\pi)$, i.e., $\mathbb{Z}^\alpha((\partial(\overline{D}(\frac{1}{2}; \frac{1}{2})) \setminus \{0\}) \subset C_{\alpha/2}$.

(2.2.5) This follows from the Jordan curve theorem.

(2.2.6) The two points of intersection, $C(0; \delta) \cap C(\frac{1}{2}; \frac{1}{2})$, are $\delta \exp(-i\theta_\delta)$ and $\delta \exp(i\theta_\delta)$, where θ_δ satisfies $\cos \theta_\delta = \delta$ (see (2.2.3)). To finish the proof, it suffices to show that there exist $\alpha \in (0, 1)$ such that $\mathbb{Z}^\alpha$ maps the arc, $\mathrm{arc}[\delta \exp(-i\theta_\delta), \delta \exp(i\theta_\delta)]$, into $\overline{D}(1; \rho)$. But

$$\begin{aligned} &\mathbb{Z}^\alpha(\mathrm{arc}[\delta \exp(-i\theta_\delta), \delta \exp(i\theta_\delta)]) \\ &= \mathrm{arc}[\delta^\alpha \exp(-i\alpha\theta_\delta), \delta \exp(i\alpha\theta_\delta)] \end{aligned}$$

Since $\delta^\alpha \nearrow 1$ as $\alpha \searrow 0$, it follows that there exists $\alpha_{\rho,\delta}$ sufficiently close to 0 such that the distance from every point on $\mathrm{arc}[\delta^\alpha \exp(-i\alpha\theta_\delta), \delta \exp(i\alpha\theta_\delta)]$ to 1 is less than ρ.

(2.2.7) This is clear from (2.2.6) plus the fact that $|\arg(z)| \leq \alpha$ for every $z \in T_\alpha$. Hence, by choosing $\alpha \leq \min\{\theta, \alpha_{\rho,\delta}\}$, the proof is complete. ☐

3. Results on linear fractional transformations (lft's)

For later use, we need the following lemma.

Lemma 3.1. *For any pair, (v_1, v_2), of distinct points on the boundary, $\mathbb{T}$, of the closed unit disk, $\overline{D}(0; 1)$, there exists a lft, f, of $\overline{D}(0; 1)$ onto $\overline{D}(0; 1)$ such that $f(v_1) = 1$ and $f(v_2) = -1$.*

Proof. If v_1 and v_2 are at opposite ends of a diameter line of $\mathbb{T}$, then a simple rotation will yield the required lft. If not, then a rotation will yield a lft taking v_1 to 1. Thus, we may assume $v_1 = 1$ and that $v_2 \in \mathbb{T} \setminus \{1, -1\}$. Henceforth, we label v_2 as v.

Any three distinct non-collinear points in $\mathbb{C}$ uniquely determines the circle passing through them. Let $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ be two sets of distinct non-collinear points on the unit circle, $\mathbb{T}$, such that the motions along $\mathbb{T}$ from z_1 to z_2 to z_3 (resp., w_1 to w_2 to w_3) are counterclockwise. By solving for w in terms of z, the cross ratio

$$\frac{w-w_1}{w_1-w_2}\frac{w_2-w_3}{w_3-w} = \frac{z-z_1}{z_1-z_2}\frac{z_2-z_3}{z_3-z}$$

uniquely determines the lft, f, that is a self-homeomorhism of $\mathbb{T}$ and $\overline{D}(0;1)$, is biholomorphic on $D(0;1)$, and maps z_i to w_i, $i = 1, 2, 3$.

We choose $\{z_1, z_2, z_3\} = \{1, v, -1\}$ and $\{w_1, w_2, w_3\} = \{1, -1, \overline{v}\}$. The cross ratio yields

$$w = f_{\mathrm{cc}}(z) = \frac{z(1-\overline{v}d) + (1+\overline{v}d)}{z(1-d) + (1+d)}, \text{ where } d = \frac{2(1+v)}{v-\overline{v}}.$$

Since it is known that the most general homeomorphism from $\overline{D}(0;1)$ onto $\overline{D}(0;1)$ and from $\mathbb{T}$ onto $\mathbb{T}$ that is conformal from $D(0;1)$ onto $D(0,1)$ and maps 1 to 1 has the form

$$f_\alpha(z) = \frac{z-\alpha}{\overline{\alpha}z-1}\frac{\overline{\alpha}-1}{1-\alpha}, \text{ for } |\alpha| < 1,$$

we must have $f = f_\alpha$ for some α with $|\alpha| < 1$. Hence, $f_\alpha(\alpha) = 0 = f(\alpha)$, from which it follows that $\alpha = -\frac{1+\overline{v}d}{1-\overline{v}d}$. □

The following facts about linear fractional transformations are basic and well known.

Lemma 3.2. *For a lft f the following conditions hold:*

(3.2.1) *The most general lft of the closed unit disk $\overline{D}(0;1)$ onto itself is $f(z) = c\frac{z-\alpha}{\overline{\alpha}z-1}$, where α and c are complex, $\left|c\right| = 1$, $\left|\alpha\right| < 1$, and $z \in \overline{D}(0;1)$.*

(3.2.2) *$f(1) = 1$ iff $c = -\frac{1-\overline{\alpha}}{1-\alpha}$ and $f(-1) = -1$ iff $c = -\frac{1+\overline{\alpha}}{1+\alpha}$.*

(3.2.3) *$f(1) = 1$ and $f(-1) = -1$ iff $c = -\frac{1-\overline{\alpha}}{1-\alpha} = -\frac{1+\overline{\alpha}}{1+\alpha}$ iff α is real.*

(3.2.4) *$u \neq 1$, $u \neq -1$, $\left|u\right| = 1$, $f(u) = u$, $f(1) = 1$ and $f(-1) = -1$ iff $\alpha = 0$ iff $f(z) = z$.*

(3.2.5) *The most general lft of the disk $\bar{D}\left(\frac{1}{2};\frac{1}{2}\right)$ onto itself that maps 0 to 0 and 1 to 1 is $g(z) = \frac{\beta z}{(1-\beta)-(1-2\beta)z}$, where $0 < \beta < 1$ (so $0 < 1-\beta < 1$ also).*

(3.2.6) *For every $w \in D\left(\frac{1}{2};\frac{1}{2}\right)$, $g(w) \to 0$ as $\beta \searrow 0$ and $g(w) \to 1$ as $\beta \nearrow 1$. These two limits are easy to verify.*

4. Established results

The following established results are central to the main theorems in this paper.

Before proceeding, recall that a *Jordan curve* is the image, $\Gamma(\mathbb{T})$, of a homeomorphism, Γ: $\mathbb{T} \mapsto \mathbb{C}$. where $\mathbb{T}$ is the unit circle. The inverse, Γ^{-1}: $\Gamma(\mathbb{T}) \mapsto \mathbb{T}$, is also a homeomorphism. We use the notation $\mathrm{int}(\Gamma(\mathbb{T}))$ to denote the bounded component of $\mathbb{C}\setminus(\Gamma(\mathbb{T}))$ and $\mathrm{ext}(\Gamma(\mathbb{T}))$ to denote the unbounded component of $\mathbb{C} \setminus (\Gamma(\mathbb{T}))$.

We now state Caratheodory's Extension theorem.

Theorem 4.1. ([3], Caratheodory's Extension Theorem for Jordan domains)

If $U \neq \mathbb{C}$ is a simply-connected, open, and connected subset of the complex plane and if ∂U is a Jordan curve, $\Gamma(\mathbb{T})$, then the Riemann map (i.e., bijective, holomorphic, and with holomorphic inverse), $f : U \mapsto D(0;1)$, from U onto $D(0;1)$ has an extension, f_{ext}, satisfying the following:

(1) $f_{ext} : \overline{U}(= U \cup \Gamma(\mathbb{T})) \mapsto \overline{D}(0;1)$ *($= D(0;1) \cup \mathbb{T}$) is a homeomorphism onto;*
(2) $f_{ext} : \Gamma(\mathbb{T}) \mapsto \mathbb{T}$ *is a homeomorphism onto;*
(3) $f_{ext} = f$ *on U and f_{ext} is conformal from U onto $D(0;1)$.*

Remark. Notice that $(f_{\text{ext}})^{-1} : \mathbb{T} \mapsto \Gamma(\mathbb{T})$ is a homeomorphism onto, but, in general, $(f_{\text{ext}})^{-1}$ and Γ are not the same function.

Corollary 4.2. *Let $\Gamma(\mathbb{T})$ be any Jordan curve, and let $z_0 \in \Gamma(\mathbb{T})$.*

Then there exists a homeomorphism $f : \Gamma(\mathbb{T}) \cup \mathrm{int}(\Gamma(\mathbb{T})) \mapsto \overline{D}(0;1)$ such that:

(1) *f restricted to $\Gamma(\mathbb{T})$ is a homeomorphism onto $\mathbb{T}$*
(2) *f restricted to $\mathrm{int}(\Gamma(\mathbb{T}))$ is a conformal map onto $D(0;1)$*
(3) *$f(z_0) = 1$.*

Proof. By Theorem 4.1, $f(z_0) \in \mathbb{T}$. There is a u with $|u| = 1$ such that $u(f(z_0)) = 1$. Thus, $F := uf$ is the required function. $\square$

Theorem 4.3. (A more general version of Caratheodory's theorem)

Let $g : D(0;1) \mapsto U$ be the inverse of the Riemann map and U is open, connected and simply connected.

Then:

g extends continuously to $G : \overline{D}(0;1)) \mapsto \overline{U})$ if and only if the boundary of U is locally connected.

Next, we state a result of Rudin.

Lemma 4.4. ([4, Lemma 15.3, p.299]) *For every set, $\{u_1, u_2, ..., u_j\} \subset \mathbb{C}$,*

$$\begin{aligned} |(1+u_1)(1+u_2)...(1+u_j)-1| &\leq (1+|u_1|)(1+|u_2|)...(1+|u_j|)-1 \\ &\leq \exp(|u_1|+|u_2|+...+|u_j|)-1. \end{aligned}$$

Next, we state and prove a property of the Euler function that plays an indispensable role in the proof of the main theorem.

Theorem 4.5. (Properties of the Euler function)

For each $0 < r < 1$, $\prod_{j=1}^{\infty}(1-z^j)$ converges uniformly on $\overline{D}(0;r)$ to a function, f, continuous on $\overline{D}(0;r)$ and holomorphic on $D(0;r)$, i.e., $f \in A(\overline{D}(0;r))$.

Proof. Since each $\prod_{j=1}^{n}(1-z^j) \in A(\overline{D}(0;r))$ and the latter is complete under the sup norm on

$\overline{D}(0;r)$, it suffices to show that $(\prod_{j=1}^{n}(1-z^j))_n$ is a Cauchy sequence in $A(\overline{D}(0;r))$.

(1) For all $k \geq 1$, $n \geq 1$, and $z \in \overline{D}(0;r)$, and using Rudin's Lemma 4.4,

$$\prod_{j=1}^{n+k}(1-z^j) - \prod_{j=1}^{n}(1-z^j) \;=\; \prod_{j=1}^{n}(1-z^j)\left[\prod_{j=n+1}^{n+k}(1-z^j)-1\right]$$

and so

$$\begin{aligned}
\left|\prod_{j=1}^{n}(1-z^j)\left[\prod_{j=n+1}^{n+k}(1-z^j)-1\right]\right| &= \prod_{j=1}^{n}|1-z^j|\left|\prod_{j=n+1}^{n+k}(1-z^j)-1\right| \\
&\leq \prod_{j=1}^{n}|1-z^j|\left[\prod_{j=n+1}^{n+k}(1+|z^j|)-1\right] \\
&\leq \prod_{j=1}^{n}|1-z^j|\left[\prod_{j=n+1}^{n+k}(1+r^j)-1\right] \\
&\leq \prod_{j=1}^{n}|1-z^j|\left[\exp(\sum_{j=n+1}^{n+k}r^j)-1\right] \\
&\leq \prod_{j=1}^{n}|1-z^j|\left[\exp(\sum_{j=n+1}^{\infty}r^j)-1\right] \\
&= \prod_{j=1}^{n}|1-z^j|\left[\exp(\frac{r^{n+1}}{1-r})-1\right]
\end{aligned}$$

(2) For all $n \geq 1$ and $z \in \overline{D}(0;r)$,

$$\begin{aligned}
|\prod_{j=1}^{n}(1-z^j)-1| &\leq \prod_{j=1}^{n}(1+|z^j|)-1 \\
&\leq \prod_{j=1}^{n}(1+r^j)-1 \\
&\leq \exp(\sum_{j=1}^{n}r^j)-1 \\
&\leq \exp(\sum_{j=1}^{\infty}r^j)-1 \\
&= \exp(\frac{r}{1-r})-1
\end{aligned}$$

Thus,

$$\begin{aligned}
|\prod_{j=1}^{n}(1-z^j)|-1 &\leq |\prod_{j=1}^{n}(1-z^j)-1| \\
&\leq \exp(\frac{r}{1-r})-1
\end{aligned}$$

so

$$|\prod_{j=1}^{n}(1-z^j)| \le \exp(\frac{r}{1-r}).$$

Putting (1) and (2) together, for all $k \ge 1$, $n \ge 1$, and $z \in \overline{D}(0;r)$,

$$|\prod_{j=1}^{n+k}(1-z^j) - \prod_{j=1}^{n}(1-z^j)| \le \exp(\frac{r}{1-r})\left[\exp(\frac{r^{n+1}}{1-r}) - 1\right] \to 0$$

as $n \to \infty$. □

Theorem 4.6. *For $z \in D(0;1)$, $\prod_{j=1}^{\infty}(1-z^j) = \exp(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j}{1-z^j})$, and for $0 < r < 1$, the series $\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j}{1-z^j}$ converges uniformly and absolutely on $\overline{D}(0;r)$.*

Proof. For the latter assertion, let $z \in \overline{D}(0;r)$, so that $|z| \le r$. By the ratio test,

$$\begin{aligned}
\frac{|a_{j+1}|}{|a_j|} &= \frac{|\frac{1}{j+1}\ \frac{z^{j+1}}{1-z^{j+1}}|}{|\frac{1}{j}\ \frac{z^j}{1-z^j}|} \\
&= \frac{j}{j+1}\frac{|1-z^j|}{|1-z^{j+1}|}|z| \\
&\le \frac{j}{j+1}\frac{|1-z^j|}{|1-z^{j+1}|}r \\
&\to r < 1 \text{ as } j \to \infty
\end{aligned}$$

Next, $\log(1-z) = -\sum_{j=1}^{\infty}\frac{z^j}{j}$ for $z \in D(0;1)$, and the series converges uniformly and absolutely on $\overline{D}(0;r)$. For all $k \ge 1$, $z \in D(0;1)$ implies $z^k \in D(0;1)$, and $z \in \overline{D}(0;r)$ implies $z^k \in \overline{D}(0;r)$, we have $\log(1-z^k) = -\sum_{j=1}^{\infty}\frac{(z^k)^j}{j}$ for $z \in D(0;1)$, and the series converges uniformly and absolutely on $\overline{D}(0;r)$.

Note that for all $z \in D(0;1)$ and $k \ge 1$, $1-z^k$ lies in the positive right half-plane.

Therefore, using the fact that $\sum_{k=1}^{\infty} z^k = \frac{z}{1-z}$ we obtain

$$\begin{aligned}
\log(\prod_{k=1}^{n}(1-z^k)) &= \sum_{k=1}^{n}\log(1-z^k) \\
&= \sum_{k=1}^{n}\left(-\sum_{j=1}^{\infty}\frac{(z^k)^j}{j}\right) \\
&= -(z+\frac{1}{2}z^2+\cdots+\frac{1}{j}z^j+\cdots) \\
&\quad -(z^2+\frac{1}{2}(z^2)^2+\cdots+\frac{1}{j}(z^2)^j+\cdots) \\
&\quad \cdots \\
&\quad -(z^n+\frac{1}{2}(z^n)^2+\cdots+\frac{1}{j}(z^n)^j+\cdots) \\
&= -(z^1+z^2+\cdots+z^n) \\
&\quad -\frac{1}{2}((z^1)^2+(z^2)^2+\cdots+(z^n)^2) \\
&\quad \cdots \\
&\quad -\frac{1}{j}((z^1)^j+(z^2)^j+\cdots+(z^n)^j) \\
&\quad \cdots
\end{aligned}$$

Using the fact that for $|z|<1$ and $j \geq 1$, $\sum_{k=1}^{n}(z^j)^k = \frac{z^j\left(1-(z^j)^n\right)}{1-z^j}$ we obtain

$$\begin{aligned}
\log(\prod_{k=1}^{n}(1-z^k)) &= -\left(\frac{z(1-z^n)}{1-z}+\frac{1}{2}\frac{z^2(1-(z^2)^n)}{1-z^2}+\frac{1}{3}\frac{z^3(1-(z^3)^n)}{1-z^3}+\cdots\right. \\
&\quad \left.+\frac{1}{j}\frac{z^j(1-(z^j)^n)}{1-z^j}+\cdots\right) \\
&= -\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j(1-(z^j)^n)}{1-z^j}
\end{aligned}$$

By the ratio test, for each $n \geq 1$, $-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j(1-(z^j)^n)}{1-z^j}$ converges uniformly and absolutely on $\overline{D}(0;r)$.

Furthermore, we claim that

$$\left\|-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j(1-(z^j)^n)}{1-z^j}-(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j}{1-z^j})\right\|_{\overline{D}(0;r)} \to 0 \text{ as } n\to\infty.$$

For $z \in \overline{D}(0;r)$,

$$\begin{aligned}\left|-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j(1-(z^j)^n)}{1-z^j}-(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j}{1-z^j})\right| &= \left|\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j}{1-z^j}z^{jn}\right| \\ &\leq \sum_{j=1}^{\infty}\frac{1}{j}\frac{|z^j|}{|1-z^j|}|z|^{jn} \\ &\leq r^n\sum_{j=1}^{\infty}\frac{1}{j}\frac{|z^j|}{|1-z^j|} \\ &\to 0 \text{ as } n\to\infty\end{aligned}$$

as claimed.

Consequently,

$$\|\exp(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j}{1-z^j})-(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j(1-(z^j)^n)}{1-z^j}\|_{\overline{D}(0;r)}\to 0 \text{ as } n\to\infty.$$

Next,

$$\exp(\log(\prod_{k=1}^{n}(1-z^k)))=\prod_{k=1}^{n}(1-z^k)=\exp\left(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j(1-(z^j)^n)}{1-z^j}\right).$$

Finally, we have

(1) $\|\prod_{j=1}^{\infty}(1-z^j)-\prod_{j=1}^{n}(1-z^j)\|_{\overline{D}(0;r)}\to 0$ as $n\to\infty$ (Theorem 4.5),

(2) $\|\exp(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j}{1-z^j})-(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j(1-(z^j)^n)}{1-z^j}\|_{\overline{D}(0;r)}\to 0$ as $n\to\infty$, and

(3) $\prod_{j=1}^{n}(1-z^j)=\exp\left(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j(1-(z^j)^n)}{1-z^j}\right)$,

which, together, imply $\prod_{j=1}^{\infty}(1-z^j)=\exp(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j}{1-z^j})$. □

Corollary 4.7. *For any* $0\leq x<1$, $\prod_{j=1}^{\infty}(1-x^j)>0$.

Proof. Since $\prod_{j=1}^{\infty}(1-x^j)=\exp(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{x^j}{1-x^j})$, and since $-\sum_{j=1}^{\infty}\frac{1}{j}\frac{x^j}{1-x^j}<0$, we have $1>\exp(-\sum_{j=1}^{\infty}\frac{1}{j}\frac{x^j}{1-x^j})>0$. □

We list various facts about circularly accessible points, topological boundary points, and kissing disks that are used throughout the paper.

Definition 4.8. Let $K\subset\mathbb{C}$ be compact, $\mathbb{C}\setminus K$ be connected, and $z\in\partial K$. Then

(1) z is a *type I boundary point* (resp., *type II*) iff each (resp., some) open nghd of z intersects (resp., does not intersect) $\mathrm{int}(K)$. We denote the set of type I (resp., type II) boundary points by ∂K_{I} (resp., ∂K_{II}). Note that $\partial K = \partial K_{\mathrm{I}} \cup \partial K_{\mathrm{II}}$ is a disjoint union.

(2) z is *circularly accessible (ca)* iff there exists $u_z \in \mathbb{T}$ and $1 > r_z > 0$ such that the closed disk $\overline{D}(c_z; r_z)$, with center $c_z = z + r_z u_z$, satisfies $\overline{D}(c_z; r_z) \cap K = \{z\}$ and $\overline{D}(c_z; r_z) \setminus \{z\} \subset \mathbb{C} \setminus K$. We denote the set of ca points by $\mathcal{P}_{\mathrm{ca}}(K)$.

(3) Such a disk is called a *kissing disk at* z, and we use the notation $(\mathrm{kd})_{z,c_z,r_z} \equiv \overline{D}(c_z; r_z)$.

Theorem 4.9. (see [2]) *$\mathcal{P}_{ca}(K)$ is dense in ∂K.*

Proof. Let $\delta/2 > 0$ and $z \in \partial K$. There exists $z_1 \in D(z; \delta/2)) \setminus K$. Since K is compact and $z_1 \notin K$, there exists $z_2 \in K$ such that $\delta_1 \equiv |z_1 - z_2| = \mathrm{dist}(z_1, K) > 0$. Note that $D(z_1; \delta_1) \subset \mathbb{C} \setminus K$ and the open line segment $(z_1, z_2) \subset \mathbb{C} \setminus K$. Clearly, since $z \in K$, $\mathrm{dist}(z_1, K) \leq |z_1 - z| < \delta/2$. Also, $|z - z_2| \leq |z_1 - z| + |z_1 - z_2| < \delta/2 + \delta/2 = \delta$. We claim that z_2 is a ca point. In fact, for every $w \in (z_1, z_2)$, $\overline{D}(w; |w - z_2|) \cap K = \{z_2\}$ and $\overline{D}(w; |w - z_2|) \setminus \{z_2\} \subset D(z_1; \delta_1) \subset \mathbb{C} \setminus K$ as is easy to see. $\square$

5. Facts about connected topological spaces

Definition 5.1. Let $(X, \mathbb{X})$ be a connected topological space, and let $\mathcal{C} \subset \mathbb{X}$ be any covering of X ($\mathbb{X}$ = topology on X).

(1) Two points x and y in X are *$\mathcal{C}$-simply n-chained* iff there exists a finite sequence, $(U_1, U_2, ..., U_n)$, of sets in $\mathcal{C}$ such that $x \in U_1$, $y \in U_n$, and $U_i \cap U_j \neq \emptyset$ iff $\left|i - j\right| \leq 1$. We term $(U_1, U_2, ..., U_n)$ a *$\mathcal{C}$-simple n-chain ($\mathcal{C}$ snc)* and we say that x is linked to y by a $\mathcal{C}$ snc.

(2) Two points x and y in X are *$\mathcal{C}$-weakly n-chained* iff there exists a finite sequence, $(U_1, U_2, ..., U_n)$, of sets in $\mathcal{C}$ such that $x \in U_1$, $y \in U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, 2, ..., n-1$. We term $(U_1, U_2, ..., U_n)$ a *$\mathcal{C}$-weak n-chain ($\mathcal{C}$ wnc)* and we say that x is linked to y by a $\mathcal{C}$ wnc.

Remark 5.2. If $(U_1, U_2, ..., U_n)$ is a $\mathcal{C}$ snc (resp., $\mathcal{C}$ wnc) that links x to y, then $(U_n, U_{n-1}, ..., U_1)$ is a $\mathcal{C}$ snc (resp., $\mathcal{C}$ wnc) that links y to x.

Theorem 5.3. *$(X, \mathbb{X})$ is a connected topological space iff for every covering, $\mathcal{C} \subset \mathbb{X}$, and every pair of distinct points x and y in X, there is a $\mathcal{C}$ wnc that links x to y.*

Proof. $\Rightarrow$

We argue the contrapositive. Suppose there exists a covering, $\mathcal{C} \subset \mathbb{X}$, and there exists a pair of points $x \neq y$ in X such that no $\mathcal{C}$ wnc links x to y. By Remark 5.2, it follows that no $\mathcal{C}$ wnc links y to x. Put

$$X' = \{x' \in X : \text{ there is a } \mathcal{C} \text{ wnc linking } x \text{ to } x'\}$$

and put

$$Y' = \{y' \in X : \text{ there is a } \mathcal{C} \text{ wnc linking } y \text{ to } y'\}.$$

Thus, y can not lie in X' and x can not lie in Y'. We now prove that both X' and Y' are open. It suffices to only show X' is open because, by symmetry, the same proof will apply to Y'.

Let $x' \in X'$ be arbitrary. By definition of X', there is a $\mathcal{C}$ wnc, $(U_1, U_2, ..., U_n)$, linking x to x', and so $x \in U_1$, $x' \in U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, 2, ..., n-1$. If we show that every point x" of U_n is in X', then since $U_n$is open, this will prove that x' is an interior point of X', and hence that X' is open. But clearly, x is linked to x" by the same $\mathcal{C}$ wnc, $(U_1, U_2, ..., U_n)$, and so x" is in X', as was to be proved.

To complete the proof, we will prove that X is the disjoint union of two non-empty open sets, and hence that X is not connected. If $X \setminus (X' \cup Y')$ is empty, we are done, so suppose $X \setminus (X' \cup Y')$ is non-empty. Let $z \in X \setminus (X' \cup Y')$. Since $\mathcal{C}$ is an open covering of X, there is an open set U in $\mathcal{C}$ such that $z \in U$. We'll show $U \subset X \setminus (X' \cup Y')$. Suppose not. Then $U \cap X' \neq \emptyset$ or $U \cap Y' \neq \emptyset$. Say that $U \cap X' \neq \emptyset$, and so there is a point $x' \in U \cap X'$. Thus, there is a $\mathcal{C}$ wnc, $(U_1, U_2, ..., U_{\mathrm{n}})$, linking x to x', and thus x is linked to z by the $\mathcal{C}$ wnc, $(U_1, U_2, ..., U_n, U)$, and so z is in X', a contradiction. Hence, z is an interior point of $X \setminus (X' \cup Y')$, and so the pair, $(X' \cup X \setminus (X' \cup Y'))$ and Y', provides a non-trivial disconnection of X, as was to be proved.

$\Leftarrow$

We argue the contrapositive. If X is disconnected, let $X = U \cup V$, where both U and V are non-empty open sets that are disjoint. Clearly, $\mathcal{C} = \{U, V\}$ is an open covering of X. If $u \in U$ and $v \in V$, then it is clear that there is no $\mathcal{C}_{\text{wnc}}$ that links u to v. ☐

Theorem 5.4. *Let $(X, \mathbb{X})$ be a connected topological space. For every $\mathcal{C}$ wnc, $(U_1, U_2, ..., U_n)$, that links x to y, where x, y is any pair of points*

in X, there is an ordered subset, $(U_1, U_{i_2}, U_{i_3}, ..., U_{i_k}, U_n)$, $1 < i_1 < i_2 < ... < i_k < n$, of $(U_1, U_2, ..., U_n)$ that is a $\mathcal{C}$ snc that links x to y.

Proof. If $n = 1$ or 2, the result is trivial, so let $n = 3$. If a $\mathcal{C}$ w3c, (U_1, U_2, U_3), is not a $\mathcal{C}$ s3c, then, clearly, we must have $U_1 \cap U_3 \neq \emptyset$. Thus, (U_1, U_3) is clearly a $\mathcal{C}$ s2c that links x to y.

We now proceed by induction. Suppose the result is true for $n = 1, 2, 3, ..., N$. Let $(U_1, U_2, ..., U_N, U_{N+1})$ be a $\mathcal{C}$ wnc that is not a $\mathcal{C}$ snc. By defintion of a $\mathcal{C}$ snc, there exists i and j in $\{1, 2, ..., N + 1\}$ such that $U_{\mathrm{i}} \cap U_{\mathrm{j}} \neq \emptyset$ and $\left|i - j\right| > 1$. We may assume that $i < j$. It follows that $(U_1, U_2, ..., U_i, U_j, ..., U_{N+1})$ is a $\mathcal{C}$ wnc that links x to y and whose length is $\leq N$, and so induction applies. □

6. Fundamental theorem on the existence of homeomorphic, conformal, norm one peaking functions for $A(K)$

Definition 6.1. Let $K \in \mathcal{K}$ with $\mathrm{int}(K) \neq \emptyset$, and let z_1 and z_2 be two distinct points in $\partial K_{\mathrm{I}} \cap \mathcal{P}_{\mathrm{ca}}(K)$.

Let Γ be a Jordan path such that

(1) $z_{\mathrm{i}} \in \Gamma$, $i = 1, 2$;
(2) $\Gamma \backslash \{z_1, z_2\} \subset \mathbb{C} \backslash K$;
(3) $\mathrm{int}(\Gamma) \subset \mathbb{C} \backslash K$.

We call such a path a kissing path with respect to z_1 and z_2.

Theorem 6.2. *Consider $K \in \mathcal{K}$ with $\mathrm{int}(K) \neq \emptyset$, and a distinct pair z_1 and z_2 in $\partial K_I \cap \mathcal{P}_{ca}(K)$.*

(a) For every such pair there exists a kissing path with respect to z_1 and z_2.

(b) There exist $f \in A(K)$ such that:

(1) $\|f\| = 1$;
(2) *f is a homeomorphism on K and conformal on int(K);*
(3) $f(z_1) = 1$*and* $f(z_2) = -1$;
(4) $f(K) \subset \{1, -1\} \cup D(0; 1)$, *f*$(K \backslash \{z_1, z_2\} \subset D(0, 1)$.

(c) There exist $g \in A(K)$ such that:

(1) $\|g\| = 1$;
(2) *g is a homeomorphism on K and conformal on int(K);*
(3) $g(z_1) = 1$*and* $g(z_2) = 0$;
(4) $g(K) \subset \{1, 0\} \cup D(\frac{1}{2}; \frac{1}{2})$, $g(K \backslash \{z_1, z_2\}) \subset D(\frac{1}{2}, \frac{1}{2})$.

Proof. Part (a)

For $i = 1, 2$, the radii, r_{z_1} and r_{z_2}, of the kissing disks, $\overline{D}(c_{z_i}; r_{z_i})$, $i = 1, 2$, may be shrunk sufficiently so that they are disjoint. Let $w_i = z_i + 2(r_{z_i})u_{z_i}$, where $u_{z_i} = \frac{c_{z_i} - z_i}{|c_{z_i} - z_i|}$, be the ends of the diameter line of the kissing disk starting at z_i and passing through the center, c_{z_i}.

It is a well-known fact that an open set $A \subset \mathbb{R}^n$ is connected iff it is pathwise-connected iff it is arcwise-connected.

Thus since $\mathbb{C} \setminus K$ is open and connected, there is a $f : [0, 1] \to \mathbb{C} \setminus K$ that is a homeomorphism onto its image and such that $f(0) = w_1$ and $f(1) = w_2$.

For $i = 1, 2$, put $S_i = f^{-1}\left(\overline{D}(c_{z_i}; r_{z_i})\right)$. Clearly $0 \in S_1$ and $1 \in S_2$. Moreover, S_i is a closed, compact proper subset of $[0, 1]$ and z_i does not lie in $f([0, 1])$.

Define $l = \sup\{t \in S_1\}$ and $u = \inf\{t \in S_2\}$. Using the definitions of supremum and infimum and the facts that f and f^{-1} are homeomorphisms, it is a straightforward consequence that:

(1) $\zeta_1 = f(l) \in \partial\overline{D}(c_{z_1}; r_{z_1}) - \{z_1\}$,

(2) $\zeta_2 = f(u) \in \partial\overline{D}(c_{z_2}; r_{z_2}) - \{z_2\}$.

Thus, if $h = f|_{[l,u]}$, then h is homeomorphic with its image, and the image is in the complement of $K \cup \overline{D}(c_{z_1}; r_{z_1}) \cup \overline{D}(c_{z_2}; r_{z_2}) - \{\zeta_1, \zeta_2\}$ and $h(l) = \zeta_1$ and $h(u) = \zeta_2$.

If we define $g(t) = f(l + (u - l)t)$, then g is a reparametrization of h. Put $G = g[0, 1]$.

There is a $\delta > 0$ sufficiently small such that the open cover, $\mathcal{C} = \{D(\zeta; \delta) : \zeta \in G\}$, of G satisfies

(1') the closure of every member of $\mathcal{C}$ is in $\mathbb{C} \setminus K$;

(2') $\overline{D}(\zeta_1; \delta) \cap \overline{D}(\zeta_2; \delta) = \emptyset$;

(3') for $i = 1, 2$, $\overline{D}(c_{z_i}; r_{z_i}) \cap \overline{D}(\zeta_i; \delta)$ is a proper subset of $\overline{D}(c_{z_i}; r_{z_i})$.

By compactness of G, there is a finite subcover, $\mathcal{C}_{\text{fin}}$ of G that includes $D(\zeta_i; \delta)$, $i = 1, ..., N$, where $\zeta_i \in G - \{\zeta_1, \zeta_2\}$ for $i = 3, ..., N$.

By Theorem 5.3, there is a $\mathcal{D}_{\text{fin}}$-weak n-chain, $\mathcal{CH}_1$, linking ζ_1 to ζ_2, and by Theorem 5.4, there is an ordered subset, $\mathcal{CH}_2 \subset \mathcal{CH}_1$ that is a $\mathcal{D}_{\text{fin}}$-simple n-chain that links ζ_1 to ζ_2. Thus $\mathcal{CH}_2 = (D(\sigma_1; \delta), ..., D(\sigma_n; \delta))$, where $\sigma_i \in G$, $i = 1, ..., n$, $\zeta_1 \in D(\sigma_1; \delta)$ and $\zeta_2 \in D(\sigma_n; \delta)$. Define the circle $C\left(\sigma; \delta = \partial\bar{D}(\sigma; \delta)\right)$. Clearly the sequence of circles, $S_{\text{circ}} = (C(\sigma_1; \delta), ..., C(\sigma_n; \delta))$, is an ordered simple n-chain.

Now $C(\sigma_1;\delta)\cap C(c_{z_1};r_{z_1})$ consists of two points: c_1^1 and c_2^1, where c_1^1 is the first intersection met when proceeding counterclockwise from z_1 along $C(c_{z_1};r_{z_1})$.

Similarly, $C(\sigma_n;\delta)\cap C(c_{z_2};r_{z_2})$ consists of two points: c_1^n and c_2^n, where c_1^n is the first intersection met when proceeding counterclockwise from z_2 along $C(c_{z_2};r_{z_2})$.

We next construct a Jordan path.

Firstly, we construct a sequence of subarcs of the circles in S_{circ}. Starting at c_1^1, proceed counterclockwise along $C(\sigma_1;\delta)$ until we reach the intersection $c_2 \in C(\sigma_2;\delta)\cap C(\sigma_1;\delta)$. Let $\text{arc}\,(1,2)$ denote the path from c_1^1 to c_2 along $C(\sigma_1;\delta)$. Proceeding inductively, we obtain a seqeunce of arc paths, joined end to end, $\text{arc}\,(1,2)$, $\text{arc}\,(2,3)$, ... , $\text{arc}\,(n-1,n)$. Note that the last arc ends at c_2^n.

Secondly, we repeat this construction by starting at c_1^n and ending at c_2^1 in order to obtain the sequence of n arcs $\text{Arc}\,(n,n-1)$, ..., $\text{Arc}\,(2,1)$.

We next construct the desired Jordan path S as follows: start at z_1, go counterclockwise along $C(c_{z_1};r_{z_1})$ to c_1^1, traverse $\text{arc}\,(1,2)$, $\text{arc}\,(2,3)$, ... , $\text{arc}\,(n-1,n)$ to c_2^n, traverse counterclockwise along $C(c_{z_2};r_{z_2})$ to c_1^n, traverse the arcs $\text{Arc}\,(n,n-1)$, ..., $\text{Arc}\,(2,1)$ to c_2^1, and the go counterclockwise along $C(c_{z_1};r_{z_1})$ to get to z_1. It is straightforward to construct (detail omitted) a homeomorphism, $\Gamma:\mathbb{T}\to S$. Thus, $S=\Gamma(\mathbb{T})$ is a Jordan curve.

Note that the winding number of $z\in\text{int}(S)$ w.r.t. Γ is $+1$. For $i=1,2$, the curve contains z_i, $\Gamma(\mathbb{T})\setminus\{z_1,z_2\}\subset\mathbb{C}\backslash K$, and $\text{int}(S)\subset\mathbb{C}\backslash K$, i.e., $\Gamma(\mathbb{T})$ is a kissing path with respect to z_1 and z_2.

Part (b)

For $z_0\in\text{int}(\Gamma(\mathbb{T}))$, define the lft, $\varphi(z)=\frac{1}{z-z_0}$, $z\in\mathbb{C}\cup\{\infty\}$ (extended complex plane). Clearly, $\varphi\circ\Gamma$ is a homeomorphism from $\mathbb{T}$ onto $(\varphi\circ\Gamma)(\mathbb{T})$ and $(\varphi\circ\Gamma)(\mathbb{T})$ Jordan curve. Furthermore,

$K\setminus\{z_1,z_2\}\subset\text{ext}((\Gamma(\mathbb{T})),$
$z_{\text{i}}\in\Gamma(\mathbb{T}),\ i=1,2,$
$\varphi(K\setminus\{z_1,z_2\})\subset\varphi(\text{ext}((\Gamma(\mathbb{T})))=\text{int}((\varphi\circ\Gamma)(\mathbb{T})),$
$w_i:=\varphi(z_i)\in(\varphi\circ\Gamma)(\mathbb{T}),\ i=1,2,$
$\varphi(\text{int}(\Gamma(\mathbb{T}))=\text{ext}((\varphi\circ\Gamma)(\mathbb{T})),$
$\partial(\text{int}((\varphi\circ\Gamma)(\mathbb{T})))=(\varphi\circ\Gamma)(\mathbb{T}).$

By Theorem 4.1, there exists $F:(\varphi\circ\Gamma)(\mathbb{T})\cup\text{int}((\varphi\circ\Gamma)(\mathbb{T}))\mapsto\overline{D}(0;1)$ such that:

(1) $F:(\varphi\circ\Gamma)(\mathbb{T})\mapsto\mathbb{T}$ is a homeomorphism onto;

(2) $F\big|_{\text{int}((\varphi\circ\Gamma)(\mathbb{T}))}$ is conformal from $\text{int}((\varphi\circ\Gamma)(\mathbb{T}))$ onto $D(0;1)$;

(3) $v_i := F(w_i) \in \mathbb{T}$.

By Lemma 3.1, there exists a lft, f, of $\overline{D}(0;1)$ onto $\overline{D}(0;1)$ such that $f(v_1) = 1$ and $f(v_2) = -1$.

Thus, the composition, $f \circ F$, satisfies the aforementioned properties of F, except that $(f \circ F)(v_1) = 1$ and $(f \circ F)(v_2) = -1$.

From the properties of all the mappings discussed, we conclude that the restriction to K of the composition map, $g := f \circ F \circ \varphi$, is an element of $P(K)$, is a homeomorphism of K into $\{1, -1\} \cup D(0;1)$, maps z_1 to 1, maps z_2 to -1, and maps $K \setminus \{z_1, z_2\}$ into $D(0,1)$.

Part (c)

By (b), select $f \in A(K)$ with all the properties listed in (b). Define $h(z) = \frac{1}{2}(1+z)$, $z \in \mathbb{C}$. Now h is a homeomorphism from $\overline{D}(0;1)$ onto $\overline{D}(\frac{1}{2};\frac{1}{2})$ and from $\mathbb{T}$ onto $\partial D(\frac{1}{2};\frac{1}{2})$, and is conformal from $D(0;1)$ onto $D(\frac{1}{2};\frac{1}{2})$. Define $g = h \circ f$. Clearly, $g \in A(K)$, and the properties under (c) clearly follow from the properties of f and h. $\square$

7. Main results

Definition 7.1. For every distinct pair of points u and v in $\partial K_{\mathrm{I}} \cap \mathcal{P}_{\mathrm{ca}}(K)$, Theorem 6.1 (b) assures the existence of an $g \in A(K)$ such that g is a homeomorphism from K onto $\overline{D}(\frac{1}{2};\frac{1}{2})$, is conformal from $\mathrm{int}(K)$ onto $D(\frac{1}{2};\frac{1}{2})$, $g(z) = 0$ iff $z = u$, $g(z) = 1$ iff $z = v$, $\|g\| = 1$, and $g(K) \subset \{0,1\} \cup D(\frac{1}{2};\frac{1}{2})$. We use the term, term, "g is a conformal homeomorphism relative to u and v".

Definition 7.2. Let $K \in \mathcal{K}$ with $\mathrm{int}(K) \neq \emptyset$, let $z_0 \in \partial K_{\mathrm{I}}$, let $(z_n)_{n\geq 1}$ be a distinct sequence in $\partial K_{\mathrm{I}} \cap \mathcal{P}_{\mathrm{ca}}(K)$ such that $|z_n - z_0| \searrow 0$, and let $(f_i)_i \subset M_{z_0} \cap A(K)$ be a sequence such that

(1) $f_i(z_j) = \delta_{i,j}$(Kronecker delta),
(2) $\|f_i\| = 1$ for all $i \geq 1$,
(3) for all $z \in K \setminus \{z_0, z_1, z_2, ..., z_j, ...\}$, $f_i(z)) \subset C_{\frac{1}{2^3}} \cap D(\frac{1}{2};\frac{1}{2})$.

We call a sequence having these properties a $(0,1)$ sequence in $A(K)$ with respect to z_0 and $(z_n)_{n\geq 1}$.

Lemma 7.3. *For every $0 < \alpha < \frac{1}{2}$, for every $0 < \zeta < \frac{1}{2}$, and for every $z_0 \in \partial K$ distinct from u and v, there exists $g_{\alpha,\zeta} \in A(K)$ such that $g_{\alpha,\zeta}$ is a conformal homeomorphism relative to u and v and $g_{\alpha,\zeta}(z_0) \in D(0;\zeta^{\frac{1}{\alpha}})$. Moreover, $f_{\alpha,\zeta} := \mathbb{Z}^\alpha \circ g_{\alpha,\zeta}$ is a conformal homeomorphism relative to u and v and $f_{\alpha,\zeta}(z_0) \in D(0;\zeta) \cap D(\frac{1}{2};\frac{1}{2})$.*

Proof. By Theorem 6.2 (c), there exists $g \in A(K)$ such that g is a conformal homeomorphism relative to u and v. Note that $g(z_0) \in D(\frac{1}{2}; \frac{1}{2})$.

Note that ζ is real, positive and $0< \zeta < \frac{1}{2}$, and that $\mathbb{Z}^{\frac{1}{\alpha}}$ is defined for all $\alpha > 0$, so $\zeta^{\frac{1}{\alpha}} = \mathbb{Z}^{\frac{1}{\alpha}}(\zeta)\searrow 0$ as $\alpha \searrow 0$.

By Lemma 3.2, part (3.2.6), there exists $\beta_{\alpha,\varsigma}$ sufficiently close to 0 such that the lft,

$F_{\alpha,\zeta}\,(z) = \frac{\beta_{\alpha,\varsigma} z}{(1 - \beta_{\alpha,\varsigma}\,) - (1 - 2\beta_{\alpha,\varsigma})z}$, $z \in \overline{D}(\frac{1}{2}; \frac{1}{2})$, satisfies $F_{\alpha,\zeta}\,(g(z_0)) \in D(0; \zeta^{\frac{1}{\alpha}})$.
Clearly, $g_{\alpha,\zeta} := F_{\alpha,\zeta} {\diamond}\, g$ is the sought-after function stated in the theorem. Finally, since $g_{\alpha,\zeta}(z_0) \in D(0; \zeta^{\frac{1}{\alpha}}) \cap D(\frac{1}{2}; \frac{1}{2})$, it has the form $g_{\alpha,\zeta}(z_0) = re^{\mathrm{i}\theta}$, where $0 < r < \zeta^{\frac{1}{\alpha}}$ and $0 \leq |\theta| < \frac{\pi}{2}$. Thus, we have $f_{\alpha,\zeta}(z_0) = \mathbb{Z}^{\alpha} \circ g_{\alpha,\zeta}(z_0) = \mathbb{Z}^{\alpha}(re^{\mathrm{i}\theta}) = r^{\alpha}e^{\mathrm{i}\alpha\theta}$, so $|f_{\alpha,\zeta}(z_0)| = |\mathbb{Z}^{\alpha}(re^{\mathrm{i}\theta}) = |r^{\alpha}e^{\mathrm{i}\alpha\theta}| = r^{\alpha} < \zeta$. Hence, $f_{\alpha,\zeta}(z_0) \in D(0; \zeta) \cap D(\frac{1}{2}; \frac{1}{2})$ as claimed. □

Lemma 7.4. *We claim that*

(a) *For every conformal homeomorphism, f, relative to u and v, for every $z_0 \in \partial K$ distinct from u and v, for every $0 < \epsilon < \frac{1}{2}$, for every $0 < \delta < \frac{1}{2}$, there exists $0 < \alpha_0 < \frac{1}{2}$ such that for every $0 < \alpha \leq \alpha_0$, $f_{\epsilon,\, \delta,\alpha} := \mathbb{Z}^{\alpha} \circ f$ is a conformal homeomorphism relative to u and v satisfying*

$$f_{\epsilon,\, \delta,\alpha}(K\backslash D(u; \epsilon)) \subset D(1; \delta).$$

(b) *In addition to the hypotheses in (a), let θ be any number such that $0\leq |\theta| < \frac{\pi}{2}$. Then there exists α_θ such that, for all α satisfying $0< \alpha \leq min\{\alpha_0,\, \alpha_\theta\}$, the function $f_{\epsilon,\, \delta,\alpha}$ in (a) also satisfies $|arg(f_{\epsilon,\, \delta,\alpha}(z)| < \theta$ for all $z \in K\backslash\, \{z_0\}$.*

Proof. (a)

Since f is a homeomorphism on K, it follows that $f(D(u; \epsilon))$ is a relatively open nhd of 0 in $f(K)$. Hence, theres exists $\rho_0 > 0$ such that the relatively open disk, $D(0; \rho_0){\cap}f(K)$, in $f(K)$ satisfies $D(0; \rho_0){\cap}f(K) \subset f(D(u; \epsilon)$. Furthermore, $f(K\backslash D(u; \epsilon)) = \overline{D}(\frac{1}{2}; \frac{1}{2})\backslash f(D(u; \epsilon))$.

By Lemma 2.2 ("teardrop" lemma), part (2.2.6), for all $0< \rho < \frac{1}{2}$ and all $0< \delta < \frac{1}{2}$, there exists $0< \alpha_{\rho,\delta} < \frac{1}{2}$ such that for all $0< \alpha \leq \alpha_{\rho,\delta}$, $\mathbb{Z}^{\alpha}(\overline{D}(\frac{1}{2}; \frac{1}{2})\backslash D(0; \rho)) \subset D(1; \delta)$.

Since $\overline{D}(\frac{1}{2}; \frac{1}{2})\backslash D(0; \rho_0) \supset \overline{D}(\frac{1}{2}; \frac{1}{2})\backslash f(D(u; \epsilon)$, we have, for all $0< \alpha \leq \alpha_{\rho_0,\delta}$,

$$\mathbb{Z}^{\alpha}(\overline{D}(\frac{1}{2}; \frac{1}{2})\backslash f(D(u; \epsilon)) \subset \mathbb{Z}^{\alpha}(\overline{D}(\frac{1}{2}; \frac{1}{2})\backslash D(0; \rho_0)) \subset D(1; \delta).$$

Setting $f_{\epsilon,\,\delta_0,\alpha} := \mathbb{Z}^{\alpha}\circ f$, we have: $f_{\epsilon,\,\delta_0,\alpha}$ is a conformal homeomorphism relative to u and v, and

$$\begin{aligned} f_{\epsilon,\,\delta,\alpha}(K\backslash D(u;\epsilon)) &= \mathbb{Z}^{\alpha}\circ f(K\backslash D(u;\epsilon)) \\ &= \mathbb{Z}^{\alpha}(f(K\backslash D(u;\epsilon))) \\ &= \mathbb{Z}^{\alpha}(\overline{D}(\frac{1}{2};\frac{1}{2})\backslash f(D(u;\epsilon)) \subset D(1;\delta). \end{aligned}$$

(b)

This follows by application of Lemma 2.2 (“teardrop”), part (2.2.7). □

Lemma 7.5. *For every distinct pair of points u and v in $\partial K_I\cap\mathcal{P}_{ca}(K)$, for every $z_0\in\partial K_I$ distinct from u and v, for every $0<\zeta<\frac{1}{2}$, for every $0<\epsilon<\frac{1}{2}$, for every $0<\delta<\frac{1}{2}$, and for every θ such that $0\leq|\theta|<\frac{\pi}{2}$, there exists a conformal homeomorphism, $f_{u,v}$, relative to u and v such that:*

(1) $\|f_{u,v}\|=1$
(2) $f_{u,v}(z)=0$ *iff* $z=u$
(3) $f_{u,v}(z)=1$ *iff* $z=v$
(4) $f_{u,v}(z_0)\in D(0;\zeta)$
(5) $f_{u,v}(K\backslash D(u;\epsilon))\subset D(1;\delta)$
(6) $|\arg(f_{u,v}(z)|<|\theta|$ *for all* $z\in K\backslash\{z_0\}$.

Proof. All of the assertions are straightforward consequences of the facts proven in Lemmas 7.3 and 7.4. □

Lemma 7.6. *Let $K\in\mathcal{K}$ and $z_0\in(\partial K)_{ni}$ (= the set of non-isolated points of K).*

Then:

Every peaking function at z_0 is non-constant, and the following are equivalent:

(A) *z_0 is a peak point for $A(K)$.*
(B) *There exists $f\in A(K)$ such that $f(z)=0$ iff $z=z_0$, $f(K)\subset\{0\}\cup D(\frac{1}{2};\frac{1}{2})$, and there exists $z\in K\backslash\{z_0\}$ such that $f(z)\in D(\frac{1}{2};\frac{1}{2})$.*
(C) *There exists $f\in A(K)$ such that $f(z)=0$ iff $z=z_0$, $f(K)\subset\{0\}\cup C_{\frac{1}{2}}$, and there exists $z\in K\backslash\{z_0\}$ such that $f(z)\in C_{\frac{1}{2}}$.*

Proof. Since z_0 is non-isolated, it is clear from the definition of peak point that every peaking function at z_0 must be non-constant.

(A)⇒(B)

There exists $g \in A(K)$ such that $g(z_0) = 1$ and $|g(z)| < 1$ for all $z \in K\setminus\{z_0\}$. Put $f = \frac{1}{2}(1-g)$. Then $f(z_0) = \frac{1}{2}(1-g(z_0)) = 0$, and for $z \in K\setminus\{z_0\}$, $|f(z) - \frac{1}{2}| = |-\frac{1}{2}g(z)| < \frac{1}{2}$, so $f(z) \in D(\frac{1}{2};\frac{1}{2})$, and thus $f(K) \subset \{0\} \cup D(\frac{1}{2};\frac{1}{2})$. Further, since g, and hence, f, are non-constant, there exists $z \in K\setminus\{z_0\}$ such that $f(z) \in D(\frac{1}{2};\frac{1}{2})$.

(B)⇒(C)

This is immediate since $\{0\} \cup D(\frac{1}{2};\frac{1}{2}) \subset \{0\} \cup C_{\frac{1}{2}}$.

(C)⇒(A)

We are given $f \in A(K)$ such that $f(z) = 0$ iff $z = z_0$ and $f(K) \subset \{0\} \cup C_{\frac{1}{2}}$. The function, $g = \frac{1}{9\|f\|} f \in A(K)$, satisfies $g(z) = 0$ iff $z = z_0$, and $g(K) \subset \{0\} \cup (D(0;\frac{1}{9}) \cap C_{\frac{1}{2}})$. Next, $h = \mathbb{Z}^{\frac{1}{2}} \circ g \in A(K)$, satisfies $h(z) = 0$ iff $z = z_0$, and $h(K) \subset \{0\} \cup (D(0;\frac{1}{3}) \cap C_{\frac{1}{4}})$. Finally, the function $F = 1 - h \in A(K)$, satisfies $\|F\| = 1$, $F(z) = 1$ iff $z = z_0$, and $|F(z)| < 1$ for all $z \in K\setminus\{z_0\}$, and so z_0 is a peak point for $A(K)$. ☐

Theorem 7.7. *Let $K \in \mathcal{K}$ with $\mathrm{int}(K) \neq \emptyset$. For every $z_0 \in \partial K_I$, for every distinct sequence, $(z_n)_{n\geq 1}$, in $\partial K_I \cap \mathcal{P}_{ca}(K)$ such that $|z_n - z_0| \searrow 0$, there exists a $(0,1)$ sequence in $A(K)$ with respect to z_0 and $(z_n)_{n\geq\mathbf{1}}$.*

Proof. In [1], Theorem 5.1.ii), it is proved that every type II boundary point of K is a peak point for $A(K)$. Therefore, we need only consider type I boundary points of K. Let $z_0 \in \partial K_{\mathrm{I}}$.

Step 1: Selection of various sequences and disks used in the proof.

1.1 Since $\mathcal{P}_{ca}(K)$ is dense in ∂K (proved in [1]), we may select any distinct sequence, $(z_n)_{n\geq 1}$, in∂K_{I} such that $|z_n - z_0| \searrow 0$.

1.2 Select a distinct positive sequence, $(\epsilon_n)_{n\geq 1}$, such that $\epsilon_n \searrow 0$ and such that the sequence, $(\overline{D}(z_n;\epsilon_n))_{n\geq 1}$, of closed disks is pairwise disjoint.

1.3 Select the sequence, $(D(1;\delta^n))_{n\geq 1}$, where $0 < \delta < \frac{1}{2}$ is fixed. Note that $\sum_{n=1,n\neq j}^{\infty} \delta^n < \sum_{n=1}^{\infty} \delta^n = \frac{\delta}{1 - \delta} < \frac{1}{2}$ for all $j \geq 1$.

1.4 Select a positive sequence, $(\theta_n)_{n\geq 1}$, where $\theta_n \searrow$, 0< $\theta_n < \frac{\pi}{2}$ and $\sum_{n=1}^{\infty} \theta_n < \frac{\pi}{8}$.

1.5 Select a distinct positive sequence, $(\zeta_n)_{n\geq 1}$, such that $\zeta_n \searrow 0$.

Step 2: Selection of various sequences of functions in $A(K)$ used in the proof.

Fix a $j \geq 1$. We shall select functions, $f_{n,j} \in A(K)$, for each $n \neq j$ so that $\mathrm{f}_{n,j}(z_j) = 1$ and $f_{n,j}(z_n) = 0$. Specifically, we use Lemma 7.5 to do this by identifying z_j with v and z_n with u. The result is the sequence, $(f_{i,n})_{n\neq j}$, in $A(K)$ such that

(1) $\|f_{n,j}\| = 1$
(2) $f_{n,j}(z) = 0$ iff $z = z_n$
(3) $f_{n,j}(z) = 1$ iff $z = z_j$
(4) $f_{n,j}(z_0) \in D(0; \zeta_n)$
(5) $f_{n,j}(K \backslash D(z_n; \epsilon_n)) \subset D(1; \delta^n)$
(6) $|\arg(f_{n,j}(z)| < |\theta_n| = \theta_n$ for all $z \in K \backslash \{z_0\}$.

Step 3: For each $j \geq 1$, construction of a sequence of finite products of the $f_{n,j}$'s.

Define the finite products, for each $j \geq 1$,

(3.1) $\Pi_p^j = \prod_{n=1, n\neq j}^{p} f_{n,j} \in A(K)$ for each $p \geq 1$.

Define the infinite sequence of finite products, for each $j \geq 1$,

(3.2) $(\Pi_p^j)_{p\geq 1}$.

Fix $j \geq 1$. For each $N \geq 1$ and all $k \geq 1$,

(3.3) $\prod_{n=1, n\neq j}^{N+k} - \prod_{n=1, n\neq j}^{N} = \prod_{n=1, n\neq j}^{N} \left[\prod_{n=N+1, n\neq j}^{N+k} -1\right]$

Step 4: Proof that for each $j \geq 1$, $(\Pi_p^j)_{p\geq 1}$ is a Cauchy sequence in $A(K)$.

Let $\epsilon > 0$. By 1.5, there exists N_ϵ such that for all $m \geq 0$, $0 < \zeta_{N_\epsilon + m} < \epsilon$. Since (4) of step 2 shows that $f_{N_\epsilon, j}(z_0) \in D(0; \zeta_{N_\epsilon})$, we have that $f^{-1}_{N_\epsilon, j}(D(0; \zeta_{N_\epsilon})$ is an open nghd of z_0. Hence, for all $z \in f^{-1}_{N_\epsilon, j}(D(0; \zeta_{N_\epsilon})$, $f_{N_\epsilon, j}(z) \in D(0; \zeta_{N_\epsilon}) \subset D(0; \epsilon)$. Therefore, for all $z \in f^{-1}_{N_\epsilon, j}(D(0; \zeta_{N_\epsilon})$, $|f_{N_\epsilon, j}(z)| < \epsilon$.

For all $k \geq 1$ and for all $z \in f^{-1}_{N_\epsilon, j}(D(0; \zeta_{N_\epsilon})$,

$$\begin{aligned}
\left|\left(\prod_{n=1,n\neq j}^{N_\epsilon+k} - \prod_{n=1,n\neq j}^{N_\epsilon}\right)(z))\right| &= \left|\left(\prod_{n=1,n\neq j}^{N_\epsilon}\right)(z)\right| \left[\left|\prod_{n=N+1,n\neq j}^{N_\epsilon+k}(z) - 1)\right|\right] \\
&= \prod_{n=1,n\neq j}^{N_\epsilon} |f_{n,j}(z)| \left|\left[\prod_{n=N+1,n\neq j}^{N_\epsilon+k} f_{n,j}(z) - 1\right]\right| \\
&= |f_{N_\epsilon,j}(z)| \prod_{n=1,n\neq j}^{N_\epsilon - 1} |f_{n,j}(z)| \left|\left[\prod_{n=N+1,n\neq j}^{N_\epsilon+k} f_{n,j}(z) - 1\right]\right| \\
&< \epsilon \left\|\left[\prod_{n=1,n\neq j}^{N_\epsilon-1} f_{n,j}\right]\left[\prod_{n=N+1,n\neq j}^{N_\epsilon+k} f_{n,j} - 1\right]\right\| \\
&< 2\epsilon.
\end{aligned}$$

Next, we let z be an arbitrary element of $K \backslash f^{-1}_{N_\epsilon, j}(D(0; \zeta_{N_\epsilon})$. Note that there exists

$D(z_0;\rho) \subset f^{-1}_{N_\epsilon,j}(D(0;\zeta_{N_\epsilon})$. Hence, $K\backslash f^{-1}_{N_\epsilon,j}(D(0;\zeta_{N_\epsilon}) \subset K\backslash D(z_0;\rho)$. By (1.2) of step 1, there exist N_ρ such that for all $p \geq N_\rho$, and all $n \geq p$, $D(z_n;\epsilon_n) \subset D(z_0;\rho)$. Thus, $K\backslash f^{-1}_{N_\epsilon,j}(D(0;\zeta_{N_\epsilon}) \subset K\backslash D(z_0;\rho) \subset K\backslash D(z_n;\epsilon_n)$ for all $n \geq p$.

Thus, for all $z \in K\backslash f^{-1}_{N_\epsilon,j}(D(0;\zeta_{N_\epsilon})$ and all $n \geq p$, $z \in K\backslash D(z_n;\epsilon_n)$, so by (5) of step 2,

$f_{n,j}(z) \in D(1;\delta^n)$, and so * $|(f_{n,j}(z) - 1| < \delta^n$ for all $n \geq p$ * .

Therefore, for $k \geq 1$,

$$\begin{aligned}
\left|\left(\prod_{n=1,n\neq j}^{p+k} - \prod_{n=1,n\neq j}^{p}\right)(z))\right| &= \left|\left(\prod_{n=1,n\neq j}^{p}\right)(z))\right|\left|\left[\prod_{n=p+1,n\neq j}^{p+k})(z) - 1\right]\right| \\
&= \prod_{n=1,n\neq j}^{p} |f_{n,j}(z)| \left|\left[\prod_{n=p+1,n\neq j}^{p+k} f_{n,j}(z) - 1\right]\right| \\
&\leq \left[\prod_{n=1,n\neq j}^{p} \|f_{n,j}(z)\|\right]\left|\left[\prod_{n=p+1,n\neq j}^{p+k} f_{n,j}(z) - 1\right]\right| \\
&\leq \left|\prod_{n=p+1,n\neq j}^{p+k} f_{n,j}(z) - 1\right| \\
&= \left|\prod_{n=p+1,n\neq j}^{p+k} [1 + (f_{n,j}(z) - 1)] - 1\right| \\
&\leq \prod_{n=p+1,n\neq j}^{p+k} [1 + |f_{n,j}(z) - 1|] - 1 \text{ (by Lemma 4.4)} \\
&\leq \exp\left(\sum_{n=p+1,n\neq j}^{p+k} |f_{n,j}(z) - 1|\right) - 1 \text{ (by Lemma 4.4)} \\
&\leq \exp\left(\sum_{n=p+1,n\neq j}^{p+k} \delta^n\right) - 1 \text{ (by * above)} \\
&\leq \exp\left(\sum_{n=p+1,n\neq j}^{\infty} \delta^n\right) - 1
\end{aligned}$$

Since $\sum_{n=p+1,n\neq j}^{\infty} \delta^n$ is the tail end of a convergent series of positive numbers, $\sum_{n=p+1,n\neq j}^{\infty} \delta^n \searrow 0$ as $p \to \infty$.

Furthermore, since $\exp(x) \searrow 1$ as $x \searrow 0$, it follows that there exist p_ϵ such that $\exp(\sum_{n=p_\epsilon+1,n\neq j}^{\infty} \delta^n) - 1 < (1+\epsilon) - 1 = \epsilon$.

Thus, for $p = p_\epsilon$, for all $z \in K \backslash f^{-1}_{N_\epsilon,j}(D(0;\zeta_{N_\epsilon})$, and for all $k \geq 1$,

$|(\prod_{n=1,n\neq j}^{p_\epsilon+k} - \prod_{n=1,n\neq j}^{p_\epsilon})(z))| < \epsilon$.

By combining this with the first part of the proof, and letting $n_\epsilon = \max\{N_\epsilon, p_\epsilon\}$, we have for all $n \geq n_\epsilon$, for all $k \geq 1$, and for all $z \in K$,
$\left|(\prod_{n=1,n\neq j}^{n_\epsilon+k} - \prod_{n=1,n\neq j}^{n_\epsilon})(z))\right| < 2\epsilon$,

and so for each $j \geq 1$, $(\Pi_p^j)_{p\geq 1}$ is a Cauchy sequence in $A(K)$ as was to be proved.

Hence, there exists $f_j \in A(K)$ such that $\|f_j - \prod_{n=1,n\neq j}^{N} f_{n,j}\| \to 0$ as $N \to \infty$.

Step 5: Properties of the f_j's.

Proof that $f_j(z_0) = 0$:

Since uniform convergence implies pointwise convergence, we have
$(\prod_{n=1,n\neq j}^{N} f_{n,j})(z) = \prod_{n=1,n\neq j}^{N} f_{n,j}(z) \to f_j(z)$ as $N \to \infty$ for all $z \in K$.
Since $z_p \to z_0$ as $p \to \infty$ and since f_j is continuous on K, $f_j(z_p) \to f_j(z_0)$ as $p \to \infty$. But by (2) of step 2, for all $n \neq j$, $f_{n,j}(z_n) = 0$. Thus, for $N_p > p$, and for all p,

$$(\prod_{n=1,n\neq j}^{N_p} f_{n,j})(z_p) = \prod_{n=1,n\neq j}^{N_p} f_{n,j}(z_p) = 0,$$

and so as $p \to \infty$, $N_p \to \infty$, and hence

$$0 = (\prod_{n=1,n\neq j}^{N_p} f_{n,j})(z_p) \to f_j(z_p)$$

so $f_j(z_p) = 0$ for all p, so $0 = f_j(z_p) \to f_j(z_0)$ as $p \to \infty$, so $f_j(z_0) = 0$.

Proof that $f_j(z_j) = 1$:

By (3) of step 2, $f_{n,j}(z_j) = 1$ for all $n \neq j$. Hence, $1 = (\prod_{n=1,n\neq j}^{N} f_{n,j})(z_j) \to f_j(z_j)$ as $N \to \infty$, so $f_j(z_j) = 1$.

Proof that for $z \in (z_n)_{n\geq 0} \backslash \{z_j\}$, $f_j(z) = 0$:

By (2) of step 2, $f_{n,j}(z_n) = 0$ for all $n \neq j$. We already have shown that $f_j(z_0) = 0$. For $p \geq 1$, $p \neq j$, and $N > p$, $0 = (\prod_{n=1,n\neq j}^{N} f_{n,j})(z_p) \to f_j(z_p)$ as $N \to \infty$. Hence, $f_j(z_p) = 0$, as was to be shown.

Proof that $\|f_j\| = 1$:

$$\begin{aligned}
\|f_j\| &= \|f_j - \prod_{n=1,n\neq j}^{N} f_{n,j} + \prod_{n=1,n\neq j}^{N} f_{n,j}\| \\
&\leq \|f_j - \prod_{n=1,n\neq j}^{N} f_{n,j}\| + \|\prod_{n=1,n\neq j}^{N} f_{n,j}\| \\
&\leq \|f_j - \prod_{n=1,n\neq j}^{N} f_{n,j}\| + \prod_{n=1,n\neq j}^{N} \|f_{n,j}\| \\
&\leq \|f_j - \prod_{n=1,n\neq j}^{N} f_{n,j}\| + 1 \\
&\to 1 \text{ as } N \to \infty,
\end{aligned}$$

so $\|f_j\| \leq 1$. But, $f_j(z_j) = 1$, so $\|f_j\| = 1$.

Proof that for all $z \in K\backslash \{z_0, z_1, z_2, ..., z_j, ...\}$, $f_i(z)) \in C_{\frac{1}{8}} \cap D(\frac{1}{2}; \frac{1}{2})$:

For every $n \neq j$ and for all $z \in K\backslash \{z_0, z_1, z_2, ..., z_j, ...\}$, $0 < |f_{n,j}(z)| < 1$.

Hence, $0 < \prod_{n=1,n\neq j}^{N} |f_{n,j}(z)| < 1$ for all N, and $\prod_{n=1,n\neq j}^{N} |f_{n,j}(z)|$ is strictly decreasing as $N \to \infty$.

Furthermore, $0 < \prod_{n=1}^{N}(1-\delta^n) < 1$ for all N, $\prod_{n=1}^{N}(1-\delta^n)$ is strictly decreasing as $N \to \infty$, and $\prod_{n=1}^{N}(1-\delta^n) \searrow \prod_{n=1}^{\infty}(1-\delta^n)$ by Corollary 4.7.

Because $z \neq z_0$, there exists a disk, $D(z_0; \rho)$ that is properly contained in K and $z \notin \overline{D}(z_0; \rho)$.

There exists N_ρ such that for all $n \geq N_\rho$, $(\overline{D}(z_n; \epsilon_n) \subset D(z_0; \rho)$. Thus $z \in (K\backslash D(z_n; \epsilon_n)$, so by (5) of step 2, $f_{n,j}(z) \in D(1; \delta^n)$. Hence, $1 - \delta^n < |f_{n,j}(z)| < 1$.

Thus, for all $k \geq 1$, $0 < \prod_{n=N_\rho}^{N_\rho+k}(1-\delta^n) < \prod_{n=N_\rho,n\neq j}^{N_\rho+k} |f_{n,j}(z)| < 1$. Multiply this inequality by $\prod_{n=1}^{N_\rho-1}(1-\delta^n)$ to get

$$0 < \prod_{n=1}^{N_\rho+k}(1-\delta^n) < \left[\prod_{n=1}^{N_\rho-1}(1-\delta^n)\right]\left[\prod_{n=N_\rho,n\neq j}^{N_\rho+k} |f_{n,j}(z)|\right] < 1,$$

and so

$$0 < \prod_{n=1}^{\infty}(1-\delta^n) < \left[\prod_{n=1}^{N_\rho-1}(1-\delta^n)\right]\left[\prod_{n=N_\rho,n\neq j}^{N_\rho+k} |f_{n,j}(z)|\right] < 1,$$

and so

$$0 < \frac{\prod_{n=1}^{\infty}(1-\delta^n)}{\left[\prod_{n=1}^{N_\rho - 1}(1-\delta^n)\right]} < \prod_{n=N_\rho, n\neq j}^{N_\rho + k} |f_{n,j}(z)| < 1.$$

Let $k \to \infty$ to get

$$0 < \frac{\prod_{n=1}^{\infty}(1-\delta^n)}{\left[\prod_{n=1}^{N_\rho - 1}(1-\delta^n)\right]} \leq \prod_{n=N_\rho, n\neq j}^{\infty} |f_{n,j}(z)| < 1.$$

Multiply this inequality by $\prod_{n=1,n\neq j}^{N_\rho - 1} |f_{n,j}(z)|$ to get

$$0 < \prod_{n=1,n\neq j}^{\infty} |f_{n,j}(z)| < \prod_{n=1,n\neq j}^{N_\rho - 1} |f_{n,j}(z)| < 1,$$

so $0 < \prod_{n=1,n\neq j}^{\infty} |f_{n,j}(z)| < 1$.

Since $\prod_{n=1,n\neq j}^{N} f_{n,j}(z) \to f_j(z)$ as $N \to \infty$, we have

$$1 > \left| \prod_{n=1,n\neq j}^{N} f_{n,j}(z) \right| = \prod_{n=1,n\neq j}^{N} |f_{n,j}(z)| \to |f_j(z)|$$

as $N \to \infty$. But

$$\prod_{n=1,n\neq j}^{N} |f_{n,j}(z)| \to \prod_{n=1,n\neq j}^{\infty} |f_{n,j}(z)| > 0,$$

so $0 < |f_j(z)| < 1$.

Now $\sum_{n=1,n\neq j}^{N} \arg(f_{n,j}(z)) = \arg(\prod_{n=1,n\neq j}^{N} f_{n,j}(z)) \to \arg(f_j(z))$ as $N \to \infty$.

By (6) of step 2, we have $-\theta_n < \arg(f_{n,j}(z)) < \theta_n$, so

$$-\sum_{n=1}^{N} \theta_n < \sum_{n=1,n\neq j}^{N} \arg(f_{n,j}(z)) < \sum_{n=1}^{N} \theta_n.$$

Since $\sum_{n=1}^{N} \theta_n < \frac{\tau}{8}$, we have $-\frac{\tau}{8} < \sum_{n=1,n\neq j}^{N} \arg(f_{n,j}(z)) < \frac{\tau}{8}$, and since

$$\arg(\prod_{n=1,n\neq j}^{N} f_{n,j}(z)) \to \arg(f_j(z))$$

as $N \to \infty$, we have $-\frac{\tau}{8} \leq \arg(f_j(z)) \leq \frac{\tau}{8}$.

Hence, $f_i(z)) \in C_{\frac{1}{8}} \cap D(\frac{1}{2}; \frac{1}{2})$ as was to be proved. $\square$

Theorem 7.8. *Let $K \in \mathcal{K}$. Every $z_0 \in \partial K$ is a peak point for $A(K)$. Hence, $\mathcal{P}(A(K)) = \partial K$.*

Proof. In [1], Theorem 5.1.ii), it is proved that every type II boundary point of K is a peak point for $A(K)$.

Therefore, we need only consider type I boundary points of K.

Let $z_0 \in \partial K_{\mathrm{I}}$.

We use the sequence of functions $(f_i)_i \subset M_{z_0} \cap A(K)$ from Theorem 7.7.

Let $(a_n)_n$ be any sequence of strictly decreasing positive numbers such that $\sum_{i=1}^{\infty} a_i = 1$.

(1) Consider the sequence, $(a_i f_i)_i$, in $A(K)$, the associated sequence, $(\sum_{i=1}^{N} a_i f_i)_N$, of partial sums in $A(K)$, and the sequence, $(\sum_{i=1}^{N} \|a_i f_i\|)_N$. With respect to the latter, $\sum_{i=1}^{N} \|a_i f_i\| \le \sum_{i=1}^{N} a_i \|f_i\| \le \sum_{i=1}^{N} a_i \to 1$ as $N \to \infty$.

In other words, $(\sum_{i=1}^{N} a_i f_i)_N$ converges absolutely, hence it converges by the well-known theorem that every absolutely convergent series in a Banach space is convergent. Thus, there exists $(\sum_{i=1}^{N} \|a_i f_i\|)_N$ such that $\|f - \sum_{i=1}^{N} a_i f_i\| \to 0$ as $N \to \infty$.

Also, for $z \in K$, $(\sum_{i=1}^{N} a_i f_i)(z) = \sum_{i=1}^{N} a_i f_i(z) \to f(z)$ as $N \to \infty$, and

$\mathrm{Re}((\sum_{i=1}^{N} a_i f_i)(z)) = \mathrm{Re}(\sum_{i=1}^{N} a_i f_i(z)) = \sum_{i=1}^{N} a_i \mathrm{Re} f_i(z) \to \mathrm{Re}(f(z))$ as $N \to \infty$.

(2) Note that $f_i(z_0) = 0$ for all i.

(3) For all $k \ge 1$, $\left(\sum_{i=1}^{N} a_i f_i\right)(z_k) \to f(z_k)$ as $N \to \infty$, But for $N > k$, $\left(\sum_{i=1}^{N} a_i f_i\right)(z_k) = a_k > 0$, so $f(z_k) > 0$, and so $f(z_k) \in C_{\frac{1}{8}} \cap D(\frac{1}{2}; \frac{1}{2})$.

(4)

$$\begin{aligned}
\|f\| &= \|(f-\sum_{i=1}^{N} a_i f_i) + \sum_{i=1}^{N} a_i f_i)\| \\
&\leq \|(f-\sum_{i=1}^{N} a_i f_i)\| + \|\sum_{i=1}^{N} a_i f_i)\| \\
&\leq \|(f-\sum_{i=1}^{N} a_i f_i)\| + \sum_{i=1}^{N} a_i \\
&< \|(f-\sum_{i=1}^{N} a_i f_i)\| + 1 \\
&\to 1 \text{ as } N \to \infty,
\end{aligned}$$

so $\|f\| \leq 1$.

(5) Let $z \in K \setminus \{z_0, z_1, z_2, ..., z_n, ...\}$. By Step 5 of Theorem 7.7, for all i, we have $f_i(z) \in C_{\frac{1}{2^3}} \cap D(\frac{1}{2}; \frac{1}{2})$, so $0 < \mathrm{Re}(f_{\mathrm{i}}(z)) < 1$. Also, $0 < a_i \mathrm{Re}(f_i(z)) < a_i < 1$.

Hence, $0 < \mathrm{Re}((\sum_{i=1}^{N} a_i f_i)(z)) = \mathrm{Re}(\sum_{i=1}^{N} a_i f_i(z)) = \sum_{i=1}^{N} a_i \mathrm{Re} f_i(z) < \sum_{i=1}^{N} a_i < 1$. But $\sum_{i=1}^{N} a_i \mathrm{Re} f_i(z)$ is strictly increasing, so $\sum_{i=1}^{N} a_i \mathrm{Re} f_i(z) \nearrow \sum_{i=1}^{\infty} a_i \mathrm{Re} f_i(z) \leq 1$. On the other hand, $\sum_{i=1}^{N} a_i \mathrm{Re} f_i(z) \to \mathrm{Re}(f(z))$ as $N \to \infty$. Thus, $0 < \mathrm{Re}(f(z)) = \sum_{i=1}^{\infty} a_i \mathrm{Re} f_i(z) \leq 1$.

Therefore, by Lemma 7.6, part (C), z_0 is a peak point for $A(K)$. $\square$

8. The coincidence, with respect to $A(K)$, of the Bishop minimal boundary, the set of peak points, the topological boundary of K, and the Shilov boundary.

Definition 8.1. $PF(A(K))$ denotes the set of peaking functions in $A(K)$, i.e., the set of all $f \in A(K)$ such that there exists $z \in K$ with $f(z) = 1$ and $|f(w)| < 1$ for $w \in K \setminus \{z\}$.

$NPF(A(K)) = A(K)PF(A(K))$ denotes the set on non-peaking functions in $A(K)$.

Clearly, $A(K) = PF(A(K)) \cup NPF(A(K))$ and the union is disjoint.

For each $f \in A(K)$, $M_f := \{z \in K : |f(z)| = \|f\|\}$ is called the maximal set for f, i.e., the closed compact subset of K on which f attains its maximum modulus.

A closed subset, $S \subset K$, is a boundary for $A(K)$ in the Shilov sense iff for each $f \in A(K)$, $M_f \cap S \neq \emptyset$.

A subset, $N \subset K$, is a boundary for $A(K)$ in the Bishop sense iff for each $f \in A(K)$, $M_f \cap N \neq \emptyset$.

Theorem 8.2. *Let $K \in \mathcal{K}$.*

(a) *The intersection, $\Gamma(A(K))$, of all subsets $S \subset K$ that are boundaries for $A(K)$ in the Shilov sense is a boundary for $A(K)$ in the sense of Shilov, and is contained in every boundary for $A(K)$ in the Shilov sense. $\Gamma(A(K))$ is called the Shilov boundary for $A(K)$.*

(b) *The class of all $N \subset K$, that are boundaries for $A(K)$ in the Bishop sense contains a smallest one, $M(A(K))$, called the Bishop minimal boundary for $A(K)$, and $M(A(K)) = \mathcal{P}(A(K))$, the set of peak points for $A(K)$.*

(c) *$\mathcal{P}(A(K)) \subset \Gamma(A(K))$, i.e., every peak point for $A(K)$ is contained in the Shilov boundary for $A(K)$.*

Proof. (a) Proofs can be found in [6, 10, 11, 12].

(b) See [9], Theorem 1.

(c) It suffices to prove that for every $S \subset K$ that is a boundary for $A(K)$ in the Shilov sense, $\mathcal{P}(A(K)) \subset S$. We argue the contrapositive: Suppose there exists an $S \subset K$ that is a boundary for $A(K)$ in the Shilov sense but that $\mathcal{P}(A(K))$ is not contained in S, i.e., there exists $z_0 \in \mathcal{P}(A(K))$ such that $z_0 \notin S$. Since z_0 is a peak point, there exists $f \in A(K)$ such that $f(z_0) = 1$ and $|f(w)| < 1$ for $w \in K \setminus \{z_0\}$. Clearly, M_f, the the maximal set for f, satisfies $M_f = \{z_0\}$, and so $M_f \cap S = \emptyset$, which contradicts the fact that S is a boundary for $A(K)$ in the Shilov sense. $\square$

Theorem 8.3. *For all $K \in \mathcal{K}$, $\partial K = \mathcal{P}(A(K)) = M(A(K)) = \Gamma(A(K))$.*

Proof. $\partial K = \mathcal{P}(A(K))$ by Theorem 7.8, and $M(A(K)) = \mathcal{P}(A(K))$ by Theorem 8.2 (b), and so

$\partial K = \mathcal{P}(A(K)) = M(A(K))$.

$\mathcal{P}(A(K)) \subset \Gamma(A(K))$by Theorem 8.2 (c).

To finish the complete proof, we now show that $\Gamma(A(K)) \subset \mathcal{P}(A(K))(= \partial K$ as just shown).

By Theorem 8.2 (a), $\Gamma(A(K))$ is contained in every boundary for $A(K)$ in the Shilov sense. Hence it suffices to show that $\mathcal{P}(A(K))$ is a boundary for $A(K)$ in the Shilov sense.

We already know that $\mathcal{P}(A(K))$ is a closed subset of K since $\partial K = \mathcal{P}(A(K))$.

For every $f \in PF(A(K))$, there exists $z_f \in \mathcal{P}(A(K) = \partial K$, and so $M_f = \{z_f\}$ and $M_f \cap \mathcal{P}(A(K) \neq \emptyset$.

Next, let $f \in NPF(A(K))$ be arbitrary. Now M_f is a non-empty subset of K. Let $z \in M_f$. If $z \in \partial K$, then $z \in \mathcal{P}(A(K)$, hence, $M_f \cap \mathcal{P}(A(K) \neq \emptyset$. On the other hand, if $z \notin \partial K$, then $z \in \mathrm{int}(K)$, and hence there exist an open, connected, simply-connected set U contained in $\mathrm{int}(K)$ to which z belongs. This means that f attains its maximum modulus (with respect to $\bar{U}$) at an interior point, $w \in U$, hence by the classical maximal modulus theorem, f has the constant value $f(w)$ on $\bar{U}$). In particular, since $\partial U \subset \partial K$, there is a point $\zeta \in \partial K = \mathcal{P}(A(K)$ such that $\zeta \in M_f$. Thus, $M_f \cap \mathcal{P}(A(K) \neq \emptyset$.

Thus, for every $f \in A(K)$, $M_f \cap \mathcal{P}(A(K) \neq \emptyset$, so $\mathcal{P}(A(K)$ is a boundary for $A(K)$ in the Shilov sense as claimed. ☐

9. The set of $z \in K$ such that there exists a bounded approximate identity in M_z coincides with ∂K.

In what follows, it suffices to consider commutative normed algebras.

Definition 9.1. Let A be a commutative normed algebra. A bounded approximate identity (bai) for A is a net $\{e_\alpha\}_{\alpha \in I} \subset A$ satisfying, for some $k > 0$ and for all $\alpha \in I$, $\|e_\alpha\| \leq k$, and satisfying $\lim_\alpha x e_\alpha = x$ for any $x \in A$.

Theorem 9.2 ([13]: Cohen factorization theorem)**.** *Let A be any commutative Banach algebra. If A has a bai, then for any $f \in A$, there exists $g, h \in A$ such that $f = gh$ and for any $\delta > 0$, h can be chosen to be in the closed ideal generated by f so that $\|h - f\| < \delta$.*

Theorem 9.3. *For all $K \in \mathcal{K}$,*

$$BAI(A(K)) := \{z \in K : \text{ there exists a bai for } M_z\} = \partial K.$$

Proof. We first show $BAI(A(K)) \subset \partial K$:

Let M_{z_0} have a bai. Suppose $z_0 \in \mathrm{int}(K)$. The function, $f(z) = z - z_0$, $z \in K$, is in M_{z_0}. By Theorem 9.2, there exists $g, h \in M_{z_0}$ such that $f = gh$. Now f, g and h are holomorphic on $\mathrm{int}(K)$, and so we can take complex derivatives at $z = z_0$ and obtain $1 = f'(z_0) = g'(z_0)h(z_0) + h'(z_0)g(z_0) = 0$, a contradiction. Thus, $z_0 \in \partial K$.

Finally, we show $\partial K \subset BAI(A(K))$:

Let $z_0 \in \partial K$ be isolated. Define $e \in M_{z_0}$ by $e(z_0) = 0$, and $e(z) = 1$ for $z \in K \backslash z_0$. For $n = 1, 2, 3, ...$, put $e_n = e$. It is clear that $(e_n)_n$ is a bai for M_{z_0}.

Let $z_0 \in \partial K$ be non-isolated. By Theorem 8.3, z_0 is a peak point for $A(K)$. By Lemma 7.6 (B), there exist $f \in A(K)$ such that $f(z) = 0$ iff $z = z_0$, $f(K) \subset \{0\} \cup D(\frac{1}{2}; \frac{1}{2})$, and there exists $z_1 \in K \setminus \{z_0\}$ such that $f(z_1) \in D(\frac{1}{2}; \frac{1}{2})$.

Select $0 < \epsilon < \frac{1}{2}$ such that for all $n \geq 1$, $\frac{\epsilon}{2^n} < |f(z_1)|$. We next define the following two sequences of nbhds of 0 and 1, respectively: $(D(0; \frac{\epsilon}{2^n})_n$ and $(D(1; \frac{\epsilon}{2^n})_n$.

By Lemma 2.2, part 2.2.6, for each $n \geq 1$, there exists $0 < \alpha_n < 1$ such that for all $0 < \alpha \leq \alpha_n$, $\mathbb{Z}^\alpha$ maps $\overline{D}(\frac{1}{2}; \frac{1}{2}) \setminus D(0; \frac{\epsilon}{2^n})$ into $D(1; \frac{\epsilon}{2^n})$. If we define $f_n = \mathbb{Z}^{\alpha_n} \circ f$ for all $n \geq 1$, then it is a straightforward verification that $(f_n)_n$ is a bai for M_{z_0}. $\square$

10. The set of $z \in K$ satisfying the Bishop $\frac{1}{4} - \frac{3}{4}$ property is equal to ∂K

Definition 10.1. We say $z \in K$ satisfies the Bishop $\frac{1}{4} - \frac{3}{4}$ property [14] iff for every open nghd U of z, there exists $f \in A(K)$ such that $\|f\|_K \leq 1$, $f(z)$ is real and $f(z) > \frac{3}{4}$, and $|f(w)| < \frac{1}{4}$ for all $w \in K \setminus U$.

Theorem 10.2. *Let $K \in \mathcal{K}$ and $z \in K$. The following are equivalent;*

10.2.1 $\quad z \in \partial K$
10.2.2 $\quad$ *z is a peak point for $A(K)$*
10.2.3 $\quad$ *z satisfies the Bishop $\frac{1}{4} - \frac{3}{4}$ property*

Proof. 10.2.1 iff 10.2.2: This is just Theorem 7.8.

10.2.2 implies 10.2.3: Let U be any open neighborhood of z. and let f be a peaking function at z. The function $g = \frac{1}{2}(1 + f)$ is also a peaking function at z with $\|g\|_K \leq 1$, $g(z) = 1 > \frac{3}{4}$. Furthermore, the image of g is contained in $\{1\} \bigcup D(\frac{1}{2}; \frac{1}{2})$. The image, $g(K \setminus U)$, is a compact subset of $\{1\} \bigcup D(\frac{1}{2}; \frac{1}{2})$ disjoint from $\{1\}$. Thus, $\sup \{|w - 0| : w \in g(K \setminus U) := \rho < 1\}$. By taking a sufficiently high power, g^n, of g, it is clear that g^n satisfies $\|g^n\|_K \leq 1$, $g^n(z) = 1 > \frac{3}{4}$ and $|g^n(w)| < \frac{1}{4}$ for all $w \in K \setminus U$ as was to be proved.

10.2.3 implies 10.2.1: We prove the contrapositive., i.e., negation of 10.2.1 implies negation of 10.2.3. Let $z \notin \partial K$. Thus $z \in \operatorname{int}(K)$. Let $U \subset \operatorname{int}K$ be an open connected neighborhood of z, e.g., the open connected component of $\operatorname{int}K$ that contains z. Suppose there exists $f \in A(K)$ such that $\|f\|_K \leq 1$, $f(z) > \frac{3}{4}$ and $|f(w)| < \frac{1}{4}$ for all $w \in K \setminus U$. By the maximum modulus theorem for U: on the one hand, $\max_{w \in \bar{U}} |f(w)| \geq f(z) > \frac{3}{4}$ while on the other $|f(w)| \leq \frac{1}{4}$ for $w \in \partial U$. Thus, z cannot satisfy the Bishop $\frac{1}{4} - \frac{3}{4}$ property. $\square$

11. The set of strong boundary points for $A(K)$ is equal to ∂K

Definition 11.1. We say that $z \in K$ is a strong boundary point for $A(K)$ iff for every open neighborhood U_z, there exists $f \in A(K)$ such that $\|f\| = f(z) = 1$ and for every $w \in K - U_z$, $|f(w)| < 1$. The set of strong boundary points for $A(K)$ is denoted $SB(A(K))$.

Theorem 11.2. *$SB(A(K)) = \partial K$.*

Proof. If $\mathrm{int}(K) = \emptyset$, then $K = (\partial K)_{\mathrm{II}} = \partial K = \mathcal{P}(A(K))$ (see (ii) under Facts in section 1). Also, $A(K) = C(K)$ by definition of $A(K)$. Thus, $SB(A(K)) \subset K = \partial K$. Conversely, if $z \in \partial K = K$, then by the properties of $C(K) = A(K)$, it is easy to see that z satisfies the conditions in the definition of a strong boundary point, hence $z \in SB(A(K))$.

If $\mathrm{int}(K) \neq \emptyset$, then we proceed as follows. Let $z \in \partial K$. By theorem 7.8, z is a peak point for $A(K)$. Clearly, by definition of a peak point, z satisfies the conditions for being a strong boundary point, so $z \in SB(A(K))$.

Conversely, let $z \in SB(A(K))$. We must show that $z \in \partial K$. Suppose not; then $z \in \mathrm{int}(K)$. From [1], Theorem 4.1, (iv), $\mathrm{int}K$ is the disjoint union of at most countably many open, connected, simply-connected sets (the "components" of $\mathrm{int}(K)$). Let U_z be the component that contains z. Select a sufficiently small closed disk such that $z \in \bar{D}(z;r) \subset U_z$. With respect to the open neighborhood, $D(z;r)$, every $f \in A(K)$ satisfying $\|f\| = f(z) = 1$ can not satisfy "$|f(w)| < 1$ for every $w \in K - D(z;r)$" because of the maximum modulus principle. Hence, $z \in \partial K$. ☐

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